PHYSICS BEYOND THE STANDARD MODEL

Exercise Sheet 8 - Solutions

Exercise 1: MSSM Higgs Masses

Ansatz:

$$\begin{aligned} H_1^1 &\equiv \frac{1}{\sqrt{2}}\Big(v_1 + R_1 + iI_1\Big) \\ H_2^2 &\equiv \frac{1}{\sqrt{2}}\Big(v_2 + R_2 + iI_2\Big) \\ m_{12}^2 &\equiv B\mu \end{aligned}$$

$$\begin{aligned} V &= m_1^2\left[\frac{1}{2}\big(v_1 + R_1 - iI_1\big)\big(v_1 + R_1 + iI_1\big) + |H_1^2|^2\right] \\ &+ m_2^2\left[\frac{1}{2}\big(v_2 + R_2 - iI_2\big)\big(v_2 + R_2 + iI_2\big) + |H_2^1|^2\right] \\ &- B\mu\left[\frac{1}{2}\big(v_1 + R_1 + iI_1\big)\big(v_2 + R_2 + iI_2\big) + \frac{1}{2}\big(v_1 + R_1 - iI_1\big)\big(v_2 + R_2 - iI_2\big) - H_1^2 H_2^1 - H_1^{2\dagger} H_2^{1\dagger}\right] \\ &+ \frac{g^2 + g'^2}{8}\left[\frac{1}{2}\big(v_1 + R_1 - iI_1\big)\big(v_1 + R_1 + iI_1\big) + |H_1^2|^2 \right. \\ &\qquad \left. -\frac{1}{2}\big(v_2 + R_2 - iI_2\big)\big(v_2 + R_2 + iI_2\big) - |H_2^1|^2\right]^2 \\ &+ \frac{g^2}{2}\left|\frac{1}{\sqrt{2}}\big(v_1 + R_1 - iI_1\big)H_2^1 + \frac{1}{\sqrt{2}}H_1^{2\dagger}\big(v_2 + R_2 + iI_2\big)\right|^2 \\ &+ \frac{\varepsilon}{2v_2^2}\left[|H_2^1|^2 + \frac{1}{2}\big(v_2 + R_2 - iI_2\big)\big(v_2 + R_2 + iI_2\big)\right]^2 \\ &= m_1^2\left[\frac{1}{2}\big(v_1^2 + 2v_1R_1 + R_1^2 + I_1^2\big) + |H_1^2|^2\right] + m_2^2\left[\frac{1}{2}\big(v_2^2 + 2v_2R_2 + R_2^2 + I_2^2\big) + |H_2^1|^2\right] \\ &- B\mu\left[v_1v_2 + v_1R_2 + v_2R_1 + R_1R_2 - I_1I_2 - H_1^2H_2^1 - H_1^{2\dagger}H_2^{1\dagger}\right] \\ &+ \frac{g^2 + g'^2}{8}\left[\frac{1}{2}\big(v_1^2 + 2v_1R_1 + R_1^2 + I_1^2\big) + |H_1^2|^2 - \frac{1}{2}\big(v_2^2 + 2v_2R_2 + R_2^2 + I_2^2\big) - |H_2^1|^2\right]^2 \\ &+ \frac{g^2}{4}\left|\big(v_1 + R_1 - iI_1\big)H_2^1 + H_1^{2\dagger}\big(v_2 + R_2 + iI_2\big)\right|^2 + \frac{\varepsilon}{2v_2^2}\left[|H_2^1|^2 + \frac{1}{2}\big(v_2^2 + 2v_2R_2 + R_2^2 + I_2^2\big)\right]^2 \end{aligned}$$

The potential is minimised for $<H_1^1>=\frac{v_1}{\sqrt{2}}$, $<H_2^2>=\frac{v_2}{\sqrt{2}}$ and $<H_1^2>=<H_2^1>=0$ (*):

$$\begin{aligned}
\left.\frac{\partial V}{\partial H_1^1}\right|_{(*)} &= \frac{\partial}{\partial H_1^1}\left[m_1^2 H_1^{1\dagger}H_1^1 - B\mu H_1^1 H_2^2 + \frac{g^2+g'^2}{8}(H_1^{1\dagger}H_1^1 - H_2^{2\dagger}H_2^2)^2\right]\Big|_{(*)} \stackrel{!}{=} 0 \\
&= \left[m_1^2 H_1^{1\dagger} - B\mu H_2^2 + \frac{g^2+g'^2}{8}2(H_1^{1\dagger}H_1^1 - H_2^{2\dagger}H_2^2)H_1^{1\dagger}\right]\Big|_{(*)} \\
&= \frac{1}{\sqrt{2}}\left[m_1^2 v_1 - B\mu v_2 + \frac{g^2+g'^2}{8}(v_1^2 - v_2^2)v_1\right] \\
\left.\frac{\partial V}{\partial H_2^2}\right|_{(*)} &= \frac{\partial}{\partial H_2^2}\left[m_2^2 H_2^{2\dagger}H_2^2 - B\mu H_1^1 H_2^2 + \frac{g^2+g'^2}{8}(H_1^{1\dagger}H_1^1 - H_2^{2\dagger}H_2^2)^2 + \frac{\varepsilon}{2v_2^2}(H_2^{2\dagger}H_2^2)^2\right]\Big|_{(*)} \stackrel{!}{=} 0 \\
&= \left[m_2^2 H_2^{2\dagger} - B\mu H_1^1 + \frac{g^2+g'^2}{8}2(H_1^{1\dagger}H_1^1 - H_2^{2\dagger}H_2^2)(-H_2^{2\dagger}) + \frac{\varepsilon}{2v_2^2}2(H_2^{2\dagger}H_2^2)H_2^{2\dagger}\right]\Big|_{(*)} \\
&= \frac{1}{\sqrt{2}}\left[m_2^2 v_2 - B\mu v_1 - \frac{g^2+g'^2}{8}(v_1^2 - v_2^2)v_2 + \frac{\varepsilon}{2}v_2\right] \\
\Longrightarrow m_1^2 &= B\mu\frac{v_2}{v_1} - \frac{g^2+g'^2}{8}v^2|\mathrm{c}^2\beta - \mathrm{s}^2\beta| \qquad = B\mu\mathrm{tg}\beta - \frac{M_Z^2}{2}|\mathrm{c}^2\beta - \mathrm{s}^2\beta| \qquad (1) \\
m_2^2 &= B\mu\frac{v_1}{v_2} + \frac{g^2+g'^2}{8}v^2|\mathrm{c}^2\beta - \mathrm{s}^2\beta| - \frac{\varepsilon}{2} \;= B\mu\mathrm{cot}\beta + \frac{M_Z^2}{2}|\mathrm{c}^2\beta - \mathrm{s}^2\beta| - \frac{\varepsilon}{2} \qquad (2)
\end{aligned}$$

1) Masses of the pseudo scalar Higgs boson A and of the Goldstone boson G from terms $\propto I_{1,2}^2, I_1 I_2$:

$$\begin{aligned}
V\big|_{I_{1,2}^2, I_1 I_2} &= \frac{1}{2}m_1^2 I_1^2 + \frac{1}{2}m_2^2 I_2^2 + B\mu I_1 I_2 + \frac{g^2+g'^2}{32}\Big[v_1^2 + I_1^2 - v_2^2 - I_2^2\Big]^2\Big|_{I_{1,2}^2, I_1 I_2} + \frac{\varepsilon}{8v_2^2}\Big[v_2^2 + I_2^2\Big]^2\Big|_{I_2^2} \\
&= \frac{1}{2}m_1^2 I_1^2 + \frac{1}{2}m_2^2 I_2^2 + B\mu I_1 I_2 + \frac{g^2+g'^2}{16}(v_1^2 - v_2^2)(I_1^2 - I_2^2) + \frac{\varepsilon}{4}I_2^2 \\
&\equiv \frac{1}{2}\begin{pmatrix} I_1 & I_2 \end{pmatrix}\mathcal{M}_a^2\begin{pmatrix} I_1 \\ I_2 \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
\mathcal{M}_a^2 &= \begin{pmatrix} m_1^2 + \frac{g^2+g'^2}{8}(v_1^2 - v_2^2) & B\mu \\ B\mu & m_2^2 + \frac{g^2+g'^2}{8}(v_2^2 - v_1^2) + \frac{\varepsilon}{2} \end{pmatrix} \qquad \text{with (1),(2)} \\
&= \begin{pmatrix} B\mu\mathrm{tg}\beta & B\mu \\ B\mu & B\mu\mathrm{cot}\beta \end{pmatrix} = B\mu\begin{pmatrix} \mathrm{tg}\beta & 1 \\ 1 & \mathrm{cot}\beta \end{pmatrix}
\end{aligned}$$

Determine the eigenvalues:

$$\begin{aligned}
& (\mathrm{tg}\beta - \lambda)(\mathrm{cot}\beta - \lambda) - 1 \stackrel{!}{=} 0 = \lambda\big[\lambda - (\mathrm{tg}\beta + \mathrm{cot}\beta)\big] \\
\Longrightarrow \lambda_1 &= 0 \qquad\qquad \Longrightarrow M_G^2 = 0 \\
\lambda_2 &= \frac{1}{\mathrm{s}\beta\mathrm{c}\beta} \qquad \Longrightarrow M_A^2 = \frac{B\mu}{\mathrm{s}\beta\mathrm{c}\beta} \quad \text{independent of } \varepsilon
\end{aligned}$$

2) Masses of the scalar Higgs bosons h, H from terms $\propto R_1^2, R_2^2, R_1R_2$:

$$\begin{aligned}
V\big|_{R_{1,2}^2,R_1R_2} &= \frac{1}{2}m_1^2R_1^2+\frac{1}{2}m_2^2R_2^2-B\mu R_1R_2 \\
&+ \frac{g^2+g'^2}{32}\Big[v_1^2+2v_1R_1+R_1^2-v_2^2-2v_2R_2-R_2^2\Big]^2\Big|_{R_{1,2}^2,R_1R_2}+\frac{\varepsilon}{8v_2^2}\Big[v_2^2+2v_2R_2+R_2^2\Big]^2\Big|_{R_2^2} \\
&= \frac{1}{2}m_1^2R_1^2+\frac{1}{2}m_2^2R_2^2-B\mu R_1R_2 \\
&\quad +\frac{g^2+g'^2}{32}\Big[6v_1^2R_1^2+6v_2^2R_2^2-2v_1^2R_2^2-2v_2^2R_1^2-8v_1v_2R_1R_2\Big]+\frac{\varepsilon}{8v_2^2}6v_2^2R_2^2 \\
&= \frac{1}{2}m_1^2R_1^2+\frac{1}{2}m_2^2R_2^2-B\mu R_1R_2 \\
&\quad +\frac{M_Z^2}{4}\Big[R_1^2(3\mathrm{c}^2\beta-\mathrm{s}^2\beta)+R_2^2(3\mathrm{s}^2\beta-\mathrm{c}^2\beta)-4R_1R_2\mathrm{s}\beta\mathrm{c}\beta\Big]+\frac{3\varepsilon}{4}R_2^2 \\
&\equiv \frac{1}{2}\begin{pmatrix}R_1 & R_2\end{pmatrix}\mathcal{M}_s^2\begin{pmatrix}R_1\\R_2\end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
\mathcal{M}_s^2 &= \begin{pmatrix} m_1^2+\frac{M_Z^2}{2}(3\mathrm{c}^2\beta-\mathrm{s}^2\beta) & -B\mu-M_Z^2\mathrm{s}\beta\mathrm{c}\beta \\ -B\mu-M_Z^2\mathrm{s}\beta\mathrm{c}\beta & m_2^2+\frac{M_Z^2}{2}(3\mathrm{s}^2\beta-\mathrm{c}^2\beta)+\frac{3\varepsilon}{2}\end{pmatrix} \text{ mit (1,2) und } M_A^2=\frac{B\mu}{\mathrm{s}\beta\mathrm{c}\beta} \\
&= \begin{pmatrix} B\mu\mathrm{tg}\beta+M_Z^2\mathrm{c}^2\beta & -(M_A^2+M_Z^2)\mathrm{s}\beta\mathrm{c}\beta \\ -(M_A^2+M_Z^2)\mathrm{s}\beta\mathrm{c}\beta & B\mu\mathrm{cot}\beta+M_Z^2\mathrm{s}^2\beta+\frac{2\varepsilon}{2}\end{pmatrix} \\
&= \begin{pmatrix} M_A^2\mathrm{s}^2\beta+M_Z^2\mathrm{c}^2\beta & -(M_A^2+M_Z^2)\mathrm{s}\beta\mathrm{c}\beta \\ -(M_A^2+M_Z^2)\mathrm{s}\beta\mathrm{c}\beta & M_A^2\mathrm{c}^2\beta+M_Z^2\mathrm{s}^2\beta+\varepsilon\end{pmatrix} \quad \mathrm{s}\beta\mathrm{c}\beta=\frac{1}{2}\mathrm{s}(2\beta) \\
&\equiv \begin{pmatrix} M_{11} & M_{12} \\ M_{12} & M_{22}\end{pmatrix}
\end{aligned}$$

Eigenvalues:
$(M_{11}-\lambda)(M_{22}-\lambda)-M_{12}^2 \stackrel{!}{=} 0$

$$\begin{aligned}
\Longrightarrow \lambda_{1,2} &= \frac{1}{2}\left(Tr\mathcal{M}_s^2 \pm \sqrt{(Tr\mathcal{M}_s^2)^2-4\det\mathcal{M}_s^2}\right) && (1)\\
&= \frac{1}{2}\left(M_{11}+M_{22}\pm\sqrt{(M_{11}-M_{22})^2+4M_{12}^2}\right), && (2)
\end{aligned}$$

where $\det\mathcal{M}_s^2 = M_{11}M_{22}-M_{12}^2$ is the determinant and $Tr\mathcal{M}_s^2 = M_{11}+M_{22}$ the trace of $\mathcal{M}_s^2$.

$$\begin{aligned}
\lambda_{1,2} &= \frac{1}{2}\Bigg(\left[M_A^2(\mathrm{s}^2\beta+\mathrm{c}^2\beta)^2+M_Z^2(\mathrm{s}^2\beta+\mathrm{c}^2\beta)+\epsilon\right] && (3)\\
&\quad \pm\sqrt{(M_A^2(\mathrm{s}^2\beta-\mathrm{c}^2\beta)+M_Z^2(\mathrm{c}^2\beta-\mathrm{s}^2\beta)-\varepsilon)^2+4\mathrm{s}^2\beta\mathrm{c}^2\beta(M_A^2+M_Z^2)^2}\Bigg) && (4)\\
&= \frac{1}{2}\left(M_A^2+M_Z^2+\varepsilon\pm\sqrt{\left[(M_Z^2-M_A^2)\cos(2\beta)-\varepsilon\right]^2+\sin^2(2\beta)(M_A^2+M_Z^2)^2}\right) && (5)
\end{aligned}$$

The masses of the two scalar Higgs bosons are maximal for $M_A \to \infty$. We obtain an upper bound by expanding the mass of the light scalar in $1/M_A^2$.

$$\begin{aligned}\lambda_2 &= \frac{1}{2}\left(M_A^2 + M_Z^2 + \varepsilon - M_A^2\sqrt{\left[\left(\frac{M_Z^2}{M_A^2} - 1\right)\cos(2\beta) - \frac{\varepsilon}{M_A^2}\right]^2 + \sin^2(2\beta)\left(1 + \frac{M_Z^2}{M_A^2}\right)^2}\right) \\ &\leq \frac{1}{2}\left(M_Z^2 + \varepsilon - \left[M_Z^2(\sin^2(2\beta) - \cos^2(2\beta)) + \varepsilon\cos(2\beta)\right]\right) = M_Z^2\cos^2(2\beta) + \varepsilon \mathrm{s}^2\beta \qquad (6)\end{aligned}$$

3) Masses of the charged Higgsbosons $H^\pm$ and of the charged Goldstone bosons $G^\pm$:
We rewrite $|H_1^2|^2 = (H_1^2)^\dagger(H_1^2)$ and $|H_2^1|^2 = (H_2^1)^\dagger(H_2^1)$ to obtain the potential of the charged Higgs bosons in matrix notation.

$$\begin{aligned}V^\pm &= \begin{pmatrix} (H_1^2)^\dagger \\ H_2^1 \end{pmatrix}\begin{pmatrix} B\mu\mathrm{tg}\beta + M_W^2\mathrm{s}^2\beta & B\mu + M_W^2 \\ B\mu + M_W^2 & B\mu\mathrm{cot}\beta + M_W^2\mathrm{c}^2\beta \end{pmatrix}\begin{pmatrix} H_1^2 \\ (H_2^1)^\dagger \end{pmatrix} \qquad (7) \\ &\equiv \begin{pmatrix} (H_1^2)^\dagger \\ H_2^1 \end{pmatrix}\mathcal{M}_\pm^2\begin{pmatrix} H_1^2 \\ (H_2^1)^\dagger \end{pmatrix}. \qquad (8)\end{aligned}$$

We can see that the mass matrix of the charged Higgses has the same form as the mass matrix of the pseudo scalar.

$$\mathcal{M}_\pm^2 = (M_A^2 + M_W^2)\mathrm{s}\beta\mathrm{c}\beta\begin{pmatrix} \mathrm{tg}\beta & 1 \\ 1 & \mathrm{cot}\beta \end{pmatrix} \qquad (9)$$

We thus find two charged massless Goldstone bosons and two charged Higgs bosons.

$$\Longrightarrow M_{G^\pm}^2 = 0, \quad M_{H^\pm}^2 = M_A^2 + M_W^2. \qquad (10)$$