Physics Beyond the Standard Model

Exercise sheet 7 - Solutions

Exercise 1: Sfermions

$$
\begin{aligned}
M^2_{\tilde{f}} &= \begin{pmatrix} m_f^2 + M^2_{\tilde{f}_L} + M_Z^2(\frac{1}{2} - e_f s_W^2)\cos 2\beta & m_f(A_f - \mu r_f) \\ m_f(A_f - \mu r_f) & m_f^2 + M^2_{\tilde{f}_R} + M_Z^2 e_f s_W^2 \cos 2\beta \end{pmatrix} \\
&\equiv \begin{pmatrix} M^2_{LL} & M^2_{LR} \\ M^2_{RL} & M^2_{RR} \end{pmatrix} \\
\mathcal{L}_{\tilde{f}} &= -\begin{pmatrix} \tilde{f}_L \\ \tilde{f}_R \end{pmatrix}^\dagger M^2_{\tilde{f}} \begin{pmatrix} \tilde{f}_L \\ \tilde{f}_R \end{pmatrix} = -\begin{pmatrix} \tilde{f}_1 \\ \tilde{f}_2 \end{pmatrix}^\dagger \begin{pmatrix} m^2_{\tilde{f}_1} & 0 \\ 0 & m^2_{\tilde{f}_2} \end{pmatrix} \begin{pmatrix} \tilde{f}_1 \\ \tilde{f}_2 \end{pmatrix}
\end{aligned}
$$

Determine the eigenvalues $m^2_{\tilde{f}_{1/2}}$:

$$
\begin{aligned}
0 &\stackrel{!}{=} \left(M^2_{LL} - m^2_{\tilde{f}_{1/2}}\right)\left(M^2_{RR} - m^2_{\tilde{f}_{1/2}}\right) - M^2_{LR}M^2_{RL} \\
&= m^4_{\tilde{f}_{1/2}} - m^2_{\tilde{f}_{1/2}}\left(M^2_{LL} + M^2_{RR}\right) + M^2_{LL}M^2_{RR} - M^2_{LR}M^2_{RL} \\
m^2_{\tilde{f}_{1/2}} &= \frac{1}{2}\left(M^2_{LL} + M^2_{RR}\right) \mp \frac{1}{2}\sqrt{\underbrace{\left(M^2_{LL} + M^2_{RR}\right)^2}_{=\left(M^2_{LL} - M^2_{RR}\right)^2} - 4\Big(M^2_{LL}M^2_{RR} - M^2_{LR}M^2_{RL}\Big)} \\
&= m_f^2 + \frac{1}{2}\left(M^2_{\tilde{f}_L} + M^2_{\tilde{f}_R} + \frac{1}{2}M_Z^2\cos 2\beta\right) \\
&\quad \mp \frac{1}{2}\sqrt{\left(M^2_{\tilde{f}_L} - M^2_{\tilde{f}_R} + \underbrace{M_Z^2\left[\frac{1}{2} - 2e_f \sin^2_W\right]\cos 2\beta}_{\text{D term}}\right)^2 + 4m_f^2\left(A_f - \mu r_f\right)^2}
\end{aligned}
$$

without the D-term we find the result from the lecture:

$$
\boxed{m^2_{\tilde{f}_{1/2}} = m_f^2 + \frac{1}{2}\left[M^2_{\tilde{f}_L} + M^2_{\tilde{f}_R} \mp \sqrt{\left(M^2_{\tilde{f}_L} - M^2_{\tilde{f}_R}\right)^2 + 4m_f^2\Big(A_f - \mu r_f\Big)^2}\right]}
$$

The chiral basis and the mass basis are related via the rotation

$$
\begin{pmatrix} \tilde{f}_1 \\ \tilde{f}_2 \end{pmatrix} = \begin{pmatrix} \cos\theta_t & \sin\theta_t \\ -\sin\theta_t & \cos\theta_t \end{pmatrix} \begin{pmatrix} \tilde{f}_L \\ \tilde{f}_R \end{pmatrix}
$$

We can determine the mixing angle θ_t:

$$\begin{aligned}
\mathcal{L}_{\tilde{f}} &= -\begin{pmatrix}\tilde{f}_1\\ \tilde{f}_2\end{pmatrix}^\dagger \begin{pmatrix} m^2_{\tilde{f}_1} & 0\\ 0 & m^2_{\tilde{f}_2}\end{pmatrix}\begin{pmatrix}\tilde{f}_1\\ \tilde{f}_2\end{pmatrix}\\
&= -\begin{pmatrix}\tilde{f}_L\\ \tilde{f}_R\end{pmatrix}^\dagger \begin{pmatrix}\cos\theta_t & -\sin\theta_t\\ \sin\theta_t & \cos\theta_t\end{pmatrix}\begin{pmatrix} m^2_{\tilde{f}_1} & 0\\ 0 & m^2_{\tilde{f}_2}\end{pmatrix}\begin{pmatrix}\cos\theta_t & \sin\theta_t\\ -\sin\theta_t & \cos\theta_t\end{pmatrix}\begin{pmatrix}\tilde{f}_L\\ \tilde{f}_R\end{pmatrix}\\
&= -\begin{pmatrix}\tilde{f}_L\\ \tilde{f}_R\end{pmatrix}^\dagger \begin{pmatrix} c^2\theta_t m^2_{\tilde{f}_1} + s^2\theta_t m^2_{\tilde{f}_2} & c\theta_t s\theta_t (m^2_{\tilde{f}_1} - m^2_{\tilde{f}_2})\\ c\theta_t s\theta_t (m^2_{\tilde{f}_1} - m^2_{\tilde{f}_2}) & s^2\theta_t m^2_{\tilde{f}_1} + c^2\theta_t m^2_{\tilde{f}_2}\end{pmatrix}\begin{pmatrix}\tilde{f}_L\\ \tilde{f}_R\end{pmatrix}
\end{aligned}$$

By comparing the coefficients,

$$\begin{aligned}
c^2\theta_t m^2_{\tilde{f}_1} + s^2\theta_t m^2_{\tilde{f}_2} &\stackrel{!}{=} m_f^2 + M^2_{\tilde{f}_L}\\
c\theta_t s\theta_t (m^2_{\tilde{f}_1} - m^2_{\tilde{f}_2}) &\stackrel{!}{=} m_f(A_f - \mu r_f)\\
s^2\theta_t m^2_{\tilde{f}_1} + c^2\theta_t m^2_{\tilde{f}_2} &\stackrel{!}{=} m_f^2 + M^2_{\tilde{f}_R}
\end{aligned}$$

we find

$$\begin{aligned}
\sin 2\theta_f &= \frac{2m_f(A_f - \mu r_f)}{m^2_{\tilde{f}_1} - m^2_{\tilde{f}_2}}\\
\cos 2\theta_f &= \frac{M^2_{\tilde{f}_L} - M^2_{\tilde{f}_R}}{m^2_{\tilde{f}_1} - m^2_{\tilde{f}_2}}
\end{aligned}$$

The mixing is proportional to the fermionmass, therefore it is only relevant for the third generation.

Exercise 2: Supersymmetry breaking in quantum mechanics

a)
$$\begin{aligned}
[\hat{\eta}, \hat{\xi}] &= \hbar[\sigma_-, \sigma_+] = \frac{\hbar}{4}[\sigma_1 - i\sigma_2, \sigma_1 + i\sigma_2]\\
&= \frac{\hbar}{4}\Big(\underbrace{[\sigma_1,\sigma_1]}_{=0} + i[\sigma_1,\sigma_2] - i[\sigma_2,\sigma_1] + \underbrace{[\sigma_2,\sigma_2]}_{=0}\Big) = \frac{2i\hbar}{4}\cdot 2i\sigma_3 = -\hbar\sigma_3\\
\Longrightarrow\ & \boxed{H = \frac{1}{2}\hat{p}^2 + V(\hat{x}) + \frac{1}{2}W(\hat{x})\hbar\sigma_3}
\end{aligned}$$

b)
$$\begin{aligned}
N_F &= \hat{\eta}\hat{\xi} = \hbar\sigma_-\sigma_+ = \frac{\hbar}{4}(\sigma_1 - i\sigma_2)(\sigma_1 + i\sigma_2) = \frac{\hbar}{4}\left(\sigma_1^2 + \sigma_2^2 + i[\sigma_1,\sigma_2]\right)\\
&= \frac{\hbar}{4}\left(2\cdot\mathbb{1} - 2\sigma_3\right) = \hbar\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}
\end{aligned}$$

Applying N_F to a bosonic/fermionic wave function:

$$\begin{matrix} N_F \left| \begin{pmatrix} \phi_1 \\ 0 \end{pmatrix} \right\rangle = 0 \left| \begin{pmatrix} \phi_1 \\ 0 \end{pmatrix} \right\rangle \\ N_F \left| \begin{pmatrix} 0 \\ \phi_2 \end{pmatrix} \right\rangle = \hbar \left| \begin{pmatrix} 0 \\ \phi_2 \end{pmatrix} \right\rangle \end{matrix} \quad \Longrightarrow \quad N_F = \text{fermion number operator}$$

c)

$$H = \frac{1}{2}\hat{p}^2 + \frac{1}{2}v^2(\hat{x}) + \frac{1}{2}\hbar v'(\hat{x})\sigma_3 \qquad v'(\hat{x}) \equiv \frac{\partial}{\partial x}v(\hat{x})$$
$$Q_i = \frac{1}{2}\sigma_i[\hat{p} + i\sigma_3 v(\hat{x})] \qquad \hat{p} = -i\hbar\frac{\partial}{\partial x}$$

$$\begin{aligned} 4[H, Q_1] &= [\hat{p}^2 + v^2 + \hbar v'\sigma_3, \sigma_1\hat{p} + i\sigma_1\sigma_3 v)] \\ &= \underbrace{i[\hat{p}^2, \sigma_1\sigma_3 v]}_{(1)} + \underbrace{[v^2, \sigma_1\hat{p}]}_{(2)} + \underbrace{\hbar[v'\sigma_3, \sigma_1\hat{p}]}_{(3)} + \underbrace{i\hbar[v'\sigma_3, \sigma_1\sigma_3 v]}_{(4)} \end{aligned}$$

$$\begin{aligned} (1) \qquad i[\hat{p}^2, \sigma_1\sigma_3 v]f &= i(-i\hbar)^2\sigma_1\sigma_3\left(\frac{\partial^2}{\partial x^2}(vf) - v\frac{\partial^2}{\partial x^2}f\right) \\ &= -i\hbar^2\sigma_1\sigma_3\left(v''f + 2v'f' + vf'' - vf''\right) \\ &= -i\hbar^2\sigma_1\sigma_3 v''f + 2\hbar\sigma_1\sigma_3 v'\hat{p}f = -\hbar^2\sigma_2 v''f - 2i\hbar\sigma_2 v'\hat{p}f \end{aligned}$$

$$\begin{aligned} (2) \quad [v^2, \sigma_1\hat{p}]f &= \sigma_1(-i\hbar)\left(v^2\frac{\partial}{\partial x}f - \frac{\partial}{\partial x}(v^2 f)\right) = -i\hbar\sigma_1\left(v^2 f' - 2vv'f - v^2 f'\right) \\ &= 2i\hbar\sigma_1 vv'f \end{aligned}$$

$$\begin{aligned} (3) \qquad \hbar[v'\sigma_3, \sigma_1\hat{p}]f &= -i\hbar^2\left(\sigma_3\sigma_1 v'\frac{\partial}{\partial x}f - \sigma_1\sigma_3\frac{\partial}{\partial x}(v'f)\right) \\ &= -i\hbar^2\left(\sigma_3\sigma_1 v'f' - \sigma_1\sigma_3 v''f - \sigma_1\sigma_3 v'f'\right) \\ &= \hbar[\sigma_3, \sigma_1]v'\hat{p}f + i\hbar^2\sigma_1\sigma_3 v''f = 2i\hbar\sigma_2 v'\hat{p}f + \hbar^2\sigma_2 v''f \end{aligned}$$

$$\begin{aligned} (4) \qquad i\hbar[v'\sigma_3, \sigma_1\sigma_3 v]f &= i\hbar\big(\sigma_3\sigma_1\sigma_3 v'vf - \sigma_1\sigma_3^2 vv'f\big) \\ &= i\hbar\big(\sigma_3(2i\sigma_2 + \sigma_3\sigma_1)v'vf - \sigma_1 vv'f\big) \\ &= i\hbar\big(2i\sigma_3\sigma_2 v'vf + \sigma_1\underbrace{[v', v]f}_{=0}\big) = -2i\hbar\sigma_1 v'vf \end{aligned}$$

$$\begin{aligned} \Longrightarrow 4[H, Q_1] &= -\hbar^2\sigma_2 v'' - 2i\hbar\sigma_2 v'\hat{p} + 2i\hbar\sigma_1 vv' + 2i\hbar\sigma_2 v'\hat{p} + \hbar^2\sigma_2 v'' - 2i\hbar\sigma_1 v'v \\ &= 0 \end{aligned}$$

The same calculation for Q_2 yields $[H, Q_2] = 0$.

$$\begin{aligned} 4\{Q_1, Q_1\} &= \{\sigma_1(\hat{p} + i\sigma_3 v), \sigma_1(\hat{p} + i\sigma_3 v)\} \\ &= \{\sigma_1\hat{p}, \sigma_1\hat{p}\} + i\{\sigma_1\hat{p}, \sigma_1\sigma_3 v\} + i\{\sigma_1\sigma_3 v, \sigma_1\hat{p}\} - \{\sigma_1\sigma_3 v, \sigma_1\sigma_3 v\} \\ &= 2\sigma_1^2\hat{p}^2 - 2(\sigma_1\sigma_3)^2 v^2 + 2i(*) \end{aligned}$$

$$
\begin{aligned}
(*)f &= \sigma_1^2\sigma_3\hat{p}vf + \sigma_1\sigma_3\sigma_1 v\hat{p}f \\
&= -i\hbar\sigma_3(v'f + vf') - i\hbar((i\sigma_2)\sigma_1 vf' \\
&= -i\hbar\sigma_3(v'f + vf') + i\hbar\sigma_3 vf' \\
&= -i\hbar\sigma_3 v'f \\
\Longrightarrow 4\{Q_1, Q_1\} &= 2\sigma_1^2\hat{p}^2 + 2\sigma_2^2 v^2 + 2\hbar\sigma_3 v' = 2\hat{p}^2 + 2v^2 + 2\hbar v'\sigma_3 = 4H
\end{aligned}
$$

Also, $\{Q_2, Q_2\} = H$.

$$
\begin{aligned}
4\{Q_1, Q_2\} &= \{\sigma_1(\hat{p} + i\sigma_3 v), \sigma_2(\hat{p} + i\sigma_3 v)\} \\
&= \underbrace{\{\sigma_1\hat{p}, \sigma_2\hat{p}\}}_{=0} + i\{\sigma_1\hat{p}, \sigma_2\sigma_3 v\} + i\{\sigma_1\sigma_3 v, \sigma_2\hat{p}\} - \underbrace{\{\sigma_1\sigma_3 v, \sigma_2\sigma_3 v\}}_{=0} \\
&= i\{\sigma_1\hat{p}, i\sigma_1 v\} + i\{-i\sigma_2 v, \sigma_2\hat{p}\} = -\{\hat{p}, v\} + \{v, \hat{p}\} = 0
\end{aligned}
$$

$$
\Longrightarrow \boxed{\begin{array}{l}[H, Q_i] = 0 \\ \{Q_i, Q_j\} = \delta_{ij} H\end{array}} \Longrightarrow \text{SUSY algebra}
$$

d)
$$
\begin{aligned}
Q_1 &= \frac{1}{2}\big[\sigma_1(\hat{p} + i\sigma_3 v)\big] = \frac{1}{2}\big[\sigma_1\hat{p} + \sigma_2 v\big] \\
&= \frac{1}{2}\begin{pmatrix} 0 & \hat{p} - iv \\ \hat{p} + iv & 0 \end{pmatrix} = -\frac{i}{2}\begin{pmatrix} 0 & \hbar\partial_x + v \\ \hbar\partial_x - v & 0 \end{pmatrix} \\
Q_2 &= \frac{1}{2}\big[\sigma_2(\hat{p} + i\sigma_3 v)\big] = \frac{1}{2}\big[\sigma_2\hat{p} - \sigma_1 v\big] \\
&= \frac{1}{2}\begin{pmatrix} 0 & -i\hat{p} - v \\ i\hat{p} - v & 0 \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} 0 & \hbar\partial_x + v \\ -\hbar\partial_x + v & 0 \end{pmatrix} = -i\sigma_3 Q_1
\end{aligned}
$$

$$
Q_1\psi(x,t) = -\frac{i}{2}\begin{pmatrix} 0 & \hbar\partial_x + v \\ \hbar\partial_x - v & 0 \end{pmatrix}\begin{pmatrix} \phi_1(x,t) \\ \phi_2(x,t) \end{pmatrix} = -\frac{i}{2}\begin{pmatrix} (\hbar\partial_x + v)\phi_2(x,t) \\ (\hbar\partial_x - v)\phi_1(x,t) \end{pmatrix}
$$

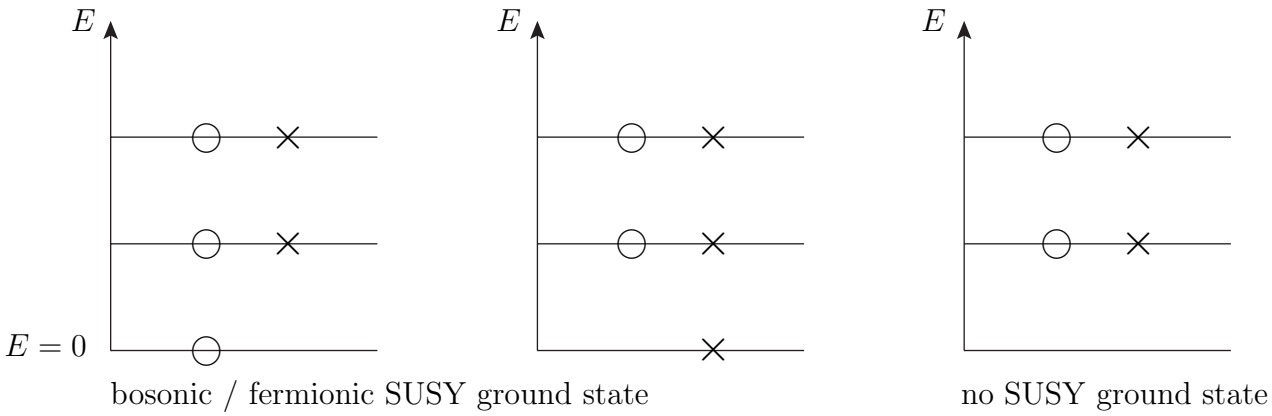
$Q_{1,2}$ are off-diagonal operators, i.e. $Q_{1,2}$ acting on a bosonic state yields a sermonic state and vice versa.

e) & f) The anticommutator $\{Q_i, Q_j\} = \delta_{ij} H$ yields $H = 2Q_i^2$. Every eigenstate of H with energy E is annihilated by $Q_{1,2}$. Conversely, every state with $Q_i \mid \psi\rangle = 0$ is an eigenstate of H with energy 0. All other states have to have positive energies because H is the square of an operator. These states are degenerate boson-fermion pairs: Let $\mid b\rangle$ be a bosonic state with $E \neq 0$. Using the SUSY algebra we can construct a fermionic state $\mid f\rangle = \sqrt{\frac{2}{E}} Q_1 \mid b\rangle = i\sqrt{\frac{2}{E}} Q_2 \mid b\rangle$. Using N_F we can show that this is a fermionic state. With $[N_F, Q_i] = i\epsilon_{ij} Q_j$ we find

$$
N_F|f\rangle = N_F\sqrt{\frac{2}{E}}Q_1|b\rangle = \sqrt{\frac{2}{E}}(i\varepsilon_{i2}Q_2 + Q_1 N_F)|b\rangle = i\sqrt{\frac{2}{E}}Q_2|b\rangle = \sqrt{\frac{2}{E}}Q_1|b\rangle = |f\rangle
$$

The spectrum of a supersymmetric quantum mechanical model has the generic form:

($\bigcirc$ bosonic state, $\times$ fermionic state)



bosonic / fermionic SUSY ground state no SUSY ground state

The eigenvalue equation for the ground state $E = 0 : \ H \mid \psi_0\rangle = 0 = 0 \mid \psi_0\rangle$ has two solutions $\psi_0^{(1,2)}$. One can show (c.f. e.g. Kalka/Soff Chapter 2.3.5) that it is impossible to normalise both solutions at the same time. Therefore, there is only one ground state for $E = 0$, which is either fermionic or bosonic. For $E > 0$, the ground state is degenerate just like all states with higher energy. Therefore, it is important to determine if a model has a supersymmetric ground state. This problem can be solved exactly for a generic quantum mechanical supersymmetric model: There is exactly one SUSY ground state if the number of zeroes of v is odd. There is no SUSY ground state if the number of zeroes of v is even.

f) The symmetry of a theory is spontaneously broken, if the ground state of the theory is not invariant under a symmetry transformation of the Hamiltonian (G is the generator of the transformation):

$$[H, G] = 0 \text{ and } \begin{cases} G|0\rangle = 0 \Longrightarrow \text{exact symmetry} \\ G|0\rangle \neq 0 \Longrightarrow \text{spontaneously broken symmetry} \end{cases}$$

Therefore, there are two possible energy spectra:

1. one (fermionic or bosonic) SUSY ground state with $E = 0$: exact SUSY

2. one (double degenerate) ground state with $E > 0$: spontaneously broken SUSY