PHYSICS BEYOND THE STANDARD MODEL

Exercise sheet 5 - Solutions

Exercise 1: Chiral Multiplet

a) We consider the transformation of a chiral superfield $\Phi(x,\theta) = \varphi(x) + \sqrt{2}\theta\psi(x) + (\theta\theta)F(x)$ under (1) and (2):

$$\begin{aligned}
\Phi(x + 2i\theta\sigma\bar{\varepsilon}; \theta + \varepsilon) &= \Phi(x;\theta) + \delta\Phi(x;\theta) \\
\delta\Phi(x;\theta) &= 2i\theta\sigma^\mu\bar{\varepsilon}\frac{\partial\Phi}{\partial x^\mu} + \varepsilon^\alpha\frac{\partial\Phi}{\partial\theta^\alpha} + \mathcal{O}(\varepsilon^2) \\
&= 2i\theta\sigma^\mu\bar{\varepsilon}\Big[\partial_\mu\varphi(x) + \sqrt{2}\theta\partial_\mu\psi(x) + (\theta\theta)\partial_\mu F(x)\Big] + \varepsilon\Big[\sqrt{2}\psi(x) + 2\theta F(x)\Big] \\
&= \sqrt{2}\varepsilon\psi(x) + 2i\theta\sigma^\mu\bar{\varepsilon}\partial_\mu\varphi(x) + 2\varepsilon\theta F(x) + 2\sqrt{2}i\theta\sigma^\mu\bar{\varepsilon}\theta\partial_\mu\psi(x) \\
&\overset{!}{=} \delta\varphi(x) + \sqrt{2}\theta\delta\psi(x) + (\theta\theta)\delta F(x)
\end{aligned}$$

$$\Longrightarrow \boxed{\begin{aligned}
\delta\varphi(x) &= \sqrt{2}\varepsilon\psi(x) \\
\delta\psi(x) &= \sqrt{2}\big[i\partial_\mu\varphi(x)\sigma^\mu\bar{\varepsilon} + \varepsilon F(x)\big] \\
\delta F(x) &= -\sqrt{2}i\partial_\mu\psi(x)\sigma^\mu\bar{\varepsilon}
\end{aligned}}$$

The Lagrangian must be invariant under SUSY up to a total derivative in order for the action to be invariant under these transformations. Consequently, the Lagrangian will only contain terms $\propto F(x)$ or $\propto F^*(x)$, i.e. coefficients of $\theta\theta$ or $\bar{\theta}\bar{\theta}$.

b) (c.f. H.Kalka/G.Soff Supersymmetry chapter 10.2.2)
The covariant derivative acting on a chiral superfield yields zero $\bar{D}\Phi = 0$. One can show that in complex superspace ($y = x + i\theta\sigma\bar{\theta}; \theta; \bar{\theta}$), this condition is equal to $\frac{\partial\Phi(y;\theta;\bar{\theta})}{\partial\theta} = 0$ $(*)$ ist. The most general chiral superfield in complex superspace is therefore of the form $\Phi(y;\theta) = \varphi(y) + \sqrt{2}\theta\psi(y) + (\theta\theta)F(y)$. In real superspace this yields

$$\begin{aligned}
\Phi(x;\theta;\bar{\theta}) &= \Phi(x + i\theta\sigma^\mu\bar{\theta};\theta) \\
&= \varphi(x) + i(\theta\sigma^\mu\bar{\theta})\partial_\mu\varphi(x) - \frac{1}{2!}(\theta\sigma^\mu\bar{\theta})(\theta\sigma^\nu\bar{\theta})\partial_\mu\partial_\nu\varphi(x) \\
&\quad + \sqrt{2}\theta\psi(x) + i\sqrt{2}(\theta\sigma^\mu\bar{\theta})\partial_\mu\theta\psi(x) + (\theta\theta)F(x) \\
&= \varphi(x) + \sqrt{2}\theta\psi(x) + (\theta\theta)F(x) \\
&\quad + i(\theta\sigma^\mu\bar{\theta})\partial_\mu\varphi(x) - \frac{i}{\sqrt{2}}(\theta\theta)\partial_\mu\big(\psi(x)\sigma^\mu\bar{\theta}\big) - \frac{1}{4}(\theta\theta)(\bar{\theta}\bar{\theta})\Box\varphi(x)
\end{aligned}$$

c) (c.f. H.Kalka/G.Soff Supersymmetry chapter 10.4.1/2/3)
We want to find a SUSY invariant Lagrangian $\mathcal{L}$, such that $S = \int d^4x\mathcal{L}$ is invariant under SUSY transformations. As we have seen in a), the Lagrangian can therefore only contain coefficients of $\theta\theta$ or $\bar{\theta}\bar{\theta}$. The product of chiral superfields is again a chiral superfield. In order to obtain fundamentally different terms, one has to multiply a chiral and an anti-chiral superfield: $\Phi^\dagger\Phi$, where $\Phi^\dagger$ contains the anti-particles and their superpartners. With that we find for the kinetic term of the Lagrangian, where $\big|_{\theta\theta\bar{\theta}\bar{\theta}}$ are the coefficients of $\theta\theta\bar{\theta}\bar{\theta}$:

$$\begin{aligned}\mathcal{L}_{kin} &= \Phi^\dagger(x;\theta;\bar{\theta})\Phi(x;\theta;\bar{\theta})\big|_{\theta\theta\bar{\theta}\bar{\theta}} = \frac{\partial^2}{\partial\theta^2}\frac{\partial^2}{\partial\bar{\theta}^2}\Phi^\dagger(x;\theta;\bar{\theta})\Phi(x;\theta;\bar{\theta}) \\ &= \int d\theta^2 d\bar{\theta}^2\Phi^\dagger(x;\theta;\bar{\theta})\Phi(x;\theta;\bar{\theta})\end{aligned}$$

In order to obtain the mass and interaction terms, we project out the terms proportional to F. We do this in complex superspace, therefore in order to obtain the Lagrangian we have to transform to real superspace.

$$\begin{aligned}\mathcal{L}_m &= -\int d^2\theta\frac{m}{2}\Phi^2(y;\theta)\Big|_{\theta\theta y\to x} + h.c. \\ \mathcal{L}_{int} &= -\int d^2\theta\frac{g}{3}\Phi^3(y;\theta)\Big|_{\theta\theta y\to x} + h.c.\end{aligned}$$

Higher terms in Φ cannot appear because already Φ^4 contains terms like $\propto \varphi^3\psi$ which make the Lagrangian unrenormalizable. ($[\mathcal{L}] \stackrel{!}{=} E^4$ with [fermions] $= E^{3/2}$, [bosons] $= E^1$ and [couplings] $\geq E^0$).
In the following we will only consider the kinetic term. We will calculate the complete Lagrangian independent of the auxiliary fields in the next exercise.

$$\begin{aligned}\Phi^\dagger(x;\theta;\bar{\theta})\Phi(x;\theta;\bar{\theta}) &= \Big[\varphi^* + \sqrt{2}\bar{\theta}\bar{\psi} + (\bar{\theta}\bar{\theta})F^* - i(\theta\sigma^\mu\bar{\theta})\partial_\mu\varphi^* \\ &\quad + \frac{i}{\sqrt{2}}(\bar{\theta}\bar{\theta})\partial_\mu(\theta\sigma^\mu\bar{\psi}) - \frac{1}{4}(\theta\theta)(\bar{\theta}\bar{\theta})\Box\varphi^*\Big]\Big[\varphi + \sqrt{2}\theta\psi \\ &\quad + (\theta\theta)F + i(\theta\sigma^\nu\bar{\theta})\partial_\nu\varphi - \frac{i}{\sqrt{2}}(\theta\theta)\partial_\nu(\psi\sigma^\nu\bar{\theta}) - \frac{1}{4}(\theta\theta)(\bar{\theta}\bar{\theta})\Box\varphi\Big]\end{aligned}$$

As we integrate over $d^2\theta d^2\bar{\theta}$, we only need to consider terms proportional to $(\theta\theta)(\bar{\theta}\bar{\theta})$:

$$\begin{aligned}&-\frac{1}{4}(\theta\theta)(\bar{\theta}\bar{\theta})(\varphi^*\Box\varphi + \varphi\Box\varphi^*) + (\theta\sigma^\mu\bar{\theta})(\theta\sigma^\nu\bar{\theta})(\partial_\mu\varphi^*)(\partial_\nu\varphi) - i(\theta\theta)(\bar{\theta}\bar{\psi})\partial_\nu(\psi\sigma^\nu\bar{\theta}) \\ &+ i(\bar{\theta}\bar{\theta})(\theta\psi)\partial_\mu(\theta\sigma^\mu\bar{\psi}) + (\theta\theta)(\bar{\theta}\bar{\theta})F^*F\end{aligned}$$

with $u\Box v = \partial_\mu(u\partial^\mu v) - (\partial_\mu u)(\partial^\mu v)$. Omitting total derivatives $\partial_\mu(u\partial^\mu v)$ we obtain for the kinetic part of the Lagrangian (c. f. H.Kalka/G.Soff Supersymmetry chapter 10.2.3 for an explicit calculation):

$$\boxed{\mathcal{L} = (\partial_\mu\varphi^*)(\partial^\mu\varphi) - \frac{i}{2}(\bar{\psi}\tilde{\sigma}^\mu\partial_\mu\psi + \psi\sigma^\mu\partial_\mu\bar{\psi}) + F^*F}$$

Exercise 2: SUSY–QED

Calculation of $W(\Phi)$ (c.f. H.Kalka/G.Soff Supersymmetry chapter 10.2.3):

$$\begin{aligned}
\Phi(y;\theta) &= \varphi(y) + \sqrt{2}\theta\psi(y) + (\theta\theta)F(y) \\
\Phi^2(y;\theta) &= \varphi^2(y) + 2\sqrt{2}\varphi(y)\theta\psi(y) + (\theta\theta)\big[2\varphi(y)F(y) - \psi(y)\psi(y)\big] \\
\Phi^3(y;\theta) &= \varphi^3(y) + 3\sqrt{2}\varphi^2(y)\theta\psi(y) + 3(\theta\theta)\big[\varphi^2(y)F(y) - \varphi(y)\psi(y)\psi(y)\big]
\end{aligned}$$

With that we obtain for the superpotential of a chiral superfield:

$$\begin{aligned}
\Longrightarrow W(\Phi) &= \frac{m}{2}\big[\varphi^2 + 2\sqrt{2}\varphi\theta\psi + (\theta\theta)(2\varphi F - \psi\psi)\big] \\
&\quad + \frac{\lambda}{6}\big[\varphi^3 + 3\sqrt{2}\varphi^2\theta\psi + 3(\theta\theta)(\varphi^2 F - \varphi\psi\psi)\big]
\end{aligned}$$

$$\begin{aligned}
\int d^6z W(\Phi) &= \int d^4x d^2\theta W(\Phi) = \int d^4x \frac{\partial^2}{\partial\theta^2} W(\Phi) \\
\frac{\partial^2}{\partial\theta^2} W(\Phi) &= \frac{m}{2}(2\varphi F - \psi\psi) + \frac{\lambda}{6} 3(\varphi^2 F - \varphi\psi\psi) \\
&= \left(m\varphi + \frac{\lambda}{2}\varphi^2\right) F - \left(\frac{m}{2} + \frac{\lambda}{2}\varphi\right)\psi\psi \qquad (1)
\end{aligned}$$

On the other hand

$$\begin{aligned}
W(\varphi) &= \frac{m}{2}\varphi^2 + \frac{\lambda}{6}\varphi^3 \implies \frac{\partial W(\varphi)}{\partial\varphi} = m\varphi + \frac{\lambda}{2}\varphi^2 \\
&\qquad\qquad \& \quad \frac{\partial^2 W(\varphi)}{\partial\varphi^2} = m + \lambda\varphi \\
\Longrightarrow F\frac{\partial W(\varphi)}{\partial\varphi} &- \frac{1}{2}\frac{\partial^2 W(\varphi)}{\partial\varphi^2}\bar{\psi}_L^c\psi_L = F\left(m\varphi + \frac{\lambda}{2}\varphi^2\right) - \frac{1}{2}(m + \lambda\varphi)\,\bar{\psi}_L^c\psi_L \qquad (2)
\end{aligned}$$

Which shows that

$$\begin{aligned}
S_W &= \int d^6z W(\Phi) + h.c \\
&= \int d^6z \left[F\left(m\varphi + \frac{\lambda}{2}\varphi^2\right) - \frac{1}{2}(m + \lambda\varphi)\,\psi\psi\right] + h.c \\
&= \int d^4x \left[F\frac{\partial W(\varphi)}{\partial\varphi} - \frac{1}{2}\frac{\partial^2 W(\varphi)}{\partial\varphi^2}\bar{\Psi}_L{}^c\Psi_L\right] + h.c.
\end{aligned}$$

The conjugate term contains the conjugate fields as well as the integration over $d^2\bar{\theta}$ rather than $d^2\theta$.