Physics Beyond the Standard Model

Exercise sheet 4

Exercise 1: Grassman numbers

- Taylor series:

$$\begin{aligned} \phi(\theta) &= \phi(0) + \phi'(0)\theta + \frac{1}{2}\phi''(0)\theta^2 + \dots \\ \{\theta,\theta\} &= 0 \quad \Rightarrow \quad \theta^2 = 0 \quad \Rightarrow \boxed{\phi(\theta) = \phi(0) + \phi'(0)\theta} \end{aligned} \tag{1}$$

- Translation invariance

$$\begin{aligned} \int_{-\infty}^{\infty} d\theta\phi(\theta) &= \int_{-\infty}^{\infty} d\theta\phi(\theta + c) \\ (1) \Rightarrow \int_{-\infty}^{\infty} d\theta[\phi(0) + \phi'(0)\theta] &= \int_{-\infty}^{\infty} d\theta[\phi(0) + \phi'(0)(\theta + c)] \\ &= \int_{-\infty}^{\infty} d\theta[\phi(0) + \phi'(0)\theta] + \underbrace{c\phi'(0)\int_{-\infty}^{\infty} d\theta}_{=0\ \forall c} \\ &\Rightarrow \boxed{\int_{-\infty}^{\infty} d\theta = 0} \end{aligned}$$

$$\begin{aligned} \int_{-\infty}^{\infty} d\theta\phi(\theta) &= \phi(0)\int_{-\infty}^{\infty} d\theta + \phi'(0)\int_{-\infty}^{\infty} \theta d\theta \stackrel{!}{=} \phi'(0)\cdot const. \\ const. &= 0 \Rightarrow \text{all integrals vanish} \\ const. &= 1 \Rightarrow \boxed{\int_{-\infty}^{\infty} \theta d\theta = 1} \end{aligned}$$

$$\begin{aligned}
\frac{\partial}{\partial\theta}\phi(\theta) &= \lim_{\varepsilon\to 0}\frac{\phi(\theta+\varepsilon)-\phi(\theta)}{\varepsilon} \\
&\overset{(1)}{=} \lim_{\varepsilon\to 0}\frac{[\phi(0)+\phi'(0)(\theta+\varepsilon)]-[\phi(0)+\phi'(0)\theta]}{\varepsilon} = \phi'(0) = \int_{-\infty}^{\infty} d\theta\phi(\theta) \\
&\Rightarrow \boxed{\int_{-\infty}^{\infty} d\theta\phi(\theta) = \frac{\partial}{\partial\theta}\phi(\theta)}
\end{aligned}$$

- $\vec{\theta} = \{\theta_1, \theta_2, \theta_3, \dots\}$

$$\begin{aligned}
\phi(\theta_1,\theta_2,\dots) &= \phi(\vec{0}) + \sum_i \frac{\partial\phi(\vec{\theta})}{\partial\theta_i}\theta_i + \sum_{i\neq j}\frac{\partial^2\phi(\vec{\theta})}{\partial\theta_i\partial\theta_j}\theta_i\theta_j + \dots \\
&= b_0 + b_i\theta_i + b_{ij}\theta_i\theta_j + \dots \\
\text{where } b_{ij}\theta_i\theta_j &= -b_{ij}\theta_j\theta_i \overset{!}{=} b_{ji}\theta_j\theta_i \\
&\Rightarrow \boxed{b_{ij} \text{ antisymmetric or zero}} \\
\int_{-\infty}^{\infty} d\theta_i &= 0 \ \forall i \\
\int_{-\infty}^{\infty} d\theta_i\theta_j &= \delta_{ij} = \frac{\partial}{\partial\theta_i}\theta_j \ \forall i,j
\end{aligned}$$

Exercise 2: Weyl spinors und Dirac bispinors

Algebra: vector space (summation of two elements, multiplication with c)
with operation $\circ$ of two elements of the algebra,
such that $m \circ n \in$ vector space.

e.g. vector space of 3dim vektoren: Summation $\surd$
Multiplication with c $\surd$
Scalar product $\notin$ VR
+ Vector product $\Rightarrow$ Algebra

Clifford algebra C_N:
N generatoren $\xi^i \dots \xi^N$ with $\{\xi^a, \xi^b\} = 2\delta^{ab}$

here the gamma matrices γ^μ are the generators: $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$

Grassman algebra Λ_N
associative, $\exists$ 1-element and generators ξ: $\{\xi^a, \xi^b\} = 0 \Rightarrow \theta^2 = 0$

We define the gamma matrices and spinors in the chiral basis rather than the Dirac basis as before. The representation of Weil spinors in the chiral basis is very simple and the action of the Projektors $I\!P_L$ and $I\!P_R$ is trivial. However, this comes at the cost of having to work with dotted and undotted spinor indices. Spinors with undotted indices live in adjoined spaces. Transformations to un-adjoined spaces are defined via the σ matrices, which carry one dotted and one undotted index. Spinors living in dual spaces are distinguished by using lower and upper indices. Indices can be raised and lowered using the ε tensor which has one lower and one upper index. The contraction of indices within one space is also performed using the ε tensor, this time with both indices raised or lowered.

$$\begin{aligned} I\!P_L &= \frac{1}{2}\left(\mathbb{1} - \begin{pmatrix} -\mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix}\right) = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & 0 \end{pmatrix} \\ I\!P_R &= \frac{1}{2}\left(\mathbb{1} + \begin{pmatrix} -\mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{1} \end{pmatrix} \end{aligned}$$

Note that on the one hand we have $\xi_\beta = \epsilon_{\beta\alpha}\xi^\alpha$, but also $\xi_\beta = \xi^\alpha \epsilon_{\alpha\beta}$. Hence, ξ and ϵ anti-commute.

a) $\bar{\psi}_1 I\!P_L \psi_2 = (\chi^\alpha, \xi^\dagger_{\dot{\alpha}})_1 \begin{pmatrix} \xi_\alpha \\ 0 \end{pmatrix}_2 = \chi_1 \cdot \xi_2$

b) $\bar{\psi}_1 I\!P_R \psi_2 = (\chi^\alpha, \xi^\dagger_{\dot{\alpha}})_1 \begin{pmatrix} 0 \\ \chi^{\dagger\dot{\alpha}} \end{pmatrix}_2 = \xi^\dagger_1 \cdot \chi^\dagger_2$

c) We first calculate the left-hand side

$$
\begin{aligned}
C &= i\gamma^0\gamma^2 = \begin{pmatrix} 0 & \sigma^0 \\ \bar{\sigma}^0 & 0 \end{pmatrix}\begin{pmatrix} 0 & i\sigma^2 \\ i\bar{\sigma}^2 & 0 \end{pmatrix} = \begin{pmatrix} i\bar{\sigma}^2 & 0 \\ 0 & i\sigma^2 \end{pmatrix} \\
\psi^c &= C\bar{\psi}^T = \begin{pmatrix} i\bar{\sigma}^2 \begin{pmatrix} \chi^1 \\ \chi^2 \end{pmatrix} \\ i\sigma^2 \begin{pmatrix} \xi^\dagger_{\dot{1}} \\ \xi^\dagger_{\dot{2}} \end{pmatrix} \end{pmatrix} = \left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \chi^1 \\ \chi^2 \end{pmatrix} \right. \\
&\qquad \left. \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} \xi^\dagger_{\dot{1}} \\ \xi^\dagger_{\dot{2}} \end{pmatrix} \right) = \begin{pmatrix} -\chi^2 \\ \chi^1 \\ \xi^\dagger_{\dot{2}} \\ -\xi^\dagger_{\dot{1}} \end{pmatrix} \\
\overline{\psi^c} &= (\psi^c)^\dagger\gamma_0 = (-\chi^{\dagger\dot{2}}, \chi^{\dagger\dot{1}}, \xi_2, -\xi_1)\begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} = \left(\xi_2, -\xi_1, -\chi^{\dagger\dot{2}}, \chi^{\dagger\dot{1}}\right) \\
&\Rightarrow \quad \overline{\psi^c_1}\not{P}_L\psi_2 = \left(\xi_2, -\xi_1, -\chi^{\dagger\dot{2}}, \chi^{\dagger\dot{1}}\right)_1 \begin{pmatrix} \xi_1 \\ \xi_2 \\ 0 \\ 0 \end{pmatrix}_2 = (\xi_2)_1(\xi_1)_2 - (\xi_1)_1(\xi_2)_2.
\end{aligned}
$$

The right-hand side yields

$$
\begin{aligned}
\xi^\alpha &= \epsilon^{\alpha\beta}\xi_\beta \quad \Rightarrow \quad \begin{pmatrix} \xi^1 \\ \xi^2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} \xi_2 \\ -\xi_1 \end{pmatrix} \\
\xi_1 \cdot \xi_2 &= (\xi^\alpha)_1(\xi_\alpha)_2 = (\xi^1, \xi^2)_1 \begin{pmatrix} \xi_2 \\ \xi_1 \end{pmatrix}_2 = (\xi^1)_1(\xi_2)_2 + (\xi^2)_1(\xi_1)_2 \\
&= (\xi_2)_1(\xi_2)_2 - (\xi_1)_1(\xi_1)_2.
\end{aligned}
$$

So that we find $\overline{\psi^c_1}\not{P}_L\psi_2 = \xi_1 \cdot \xi_2$.

d) We use the previous expression for $\overline{\psi^c_1}$ and find for the left-hand side

$$
\overline{\psi^c_1}\not{P}_R\psi_2 = \left(\xi_2, -\xi_1, -\chi^{\dagger\dot{2}}, \chi^{\dagger\dot{1}}\right)_1 \begin{pmatrix} 0 \\ 0 \\ \chi^{\dagger\dot{1}} \\ \chi^{\dagger\dot{2}} \end{pmatrix}_2 = -(\chi^{\dagger\dot{2}})_1(\chi^{\dagger\dot{1}})_2 + (\chi^{\dagger\dot{1}})_1(\chi^{\dagger\dot{2}})_2.
$$

The right-hand side yields

$$\chi^{\dagger\dot{\alpha}} = \epsilon^{\dot{\alpha}\dot{\beta}}\chi^\dagger_{\dot{\beta}} \quad \Rightarrow \quad \begin{pmatrix} \chi^{\dagger\dot{1}} \\ \chi^{\dagger\dot{2}} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \chi^\dagger_{\dot{1}} \\ \chi^\dagger_{\dot{2}} \end{pmatrix} = \begin{pmatrix} -\chi^\dagger_{\dot{2}} \\ \chi^\dagger_{\dot{1}} \end{pmatrix}$$

$$\begin{aligned} \chi_1^\dagger \cdot \xi_2^\dagger &= (\chi^\dagger_{\dot{\alpha}})_1(\chi^{\dagger\dot{\alpha}})_2 = (\chi^\dagger_{\dot{1}}, \chi^\dagger_{\dot{2}})_1 \begin{pmatrix} \chi^{\dagger\dot{1}} \\ \chi^{\dagger\dot{2}} \end{pmatrix}_2 = (\chi^\dagger_{\dot{1}})_1(\chi^{\dagger\dot{1}})_2 + (\chi^\dagger_{\dot{2}})_1(\chi^{\dagger\dot{2}})_2 \\ &= (-\chi^{\dagger\dot{2}})_1(\chi^{\dagger\dot{1}})_2 + (\chi^{\dagger\dot{1}})_1(\chi^{\dagger\dot{2}})_2. \end{aligned}$$

So that $\overline{\psi_1^c}I\!P_R\psi_2 = \chi_1^\dagger \cdot \xi_2^\dagger$.

e) This part is easy, since we don't have to explicitly write out components:

$$\begin{aligned} \overline{\psi_1}\gamma^\mu I\!P_L\psi_2 &= \left(\chi^\alpha, \xi^\dagger_{\dot{\alpha}}\right)_1 \begin{pmatrix} 0 & \sigma^\mu_{\alpha\dot{\beta}} \\ \bar{\sigma}^{\mu\dot{\alpha}\beta} & 0 \end{pmatrix} \begin{pmatrix} \xi_\beta \\ 0 \end{pmatrix}_2 \\ &= \left(\chi^\alpha, \xi^\dagger_{\dot{\alpha}}\right)_1 \begin{pmatrix} 0 \\ \bar{\sigma}^{\mu\dot{\alpha}\beta}\xi_\beta \end{pmatrix}_2 = (\xi^\dagger_{\dot{\alpha}})_1\bar{\sigma}^{\mu\dot{\alpha}\beta}(\xi_\beta)_2 \\ &= \xi_1^\dagger \cdot \bar{\sigma}^\mu \cdot \xi_2. \end{aligned}$$

f) The first identity can be shown in the same way as e)

$$\begin{aligned} \overline{\psi_1}\gamma^\mu I\!P_R\psi_2 &= \left(\chi^\alpha, \xi^\dagger_{\dot{\alpha}}\right)_1 \begin{pmatrix} 0 & \sigma^\mu_{\alpha\dot{\beta}} \\ \bar{\sigma}^{\mu\dot{\alpha}\beta} & 0 \end{pmatrix} \begin{pmatrix} 0 \\ \chi^{\dagger\dot{\beta}} \end{pmatrix}_2 \\ &= \left(\chi^\alpha, \xi^\dagger_{\dot{\alpha}}\right)_1 \begin{pmatrix} \sigma^\mu_{\alpha\dot{\beta}}\chi^{\dagger\dot{\beta}} \\ 0 \end{pmatrix}_2 = (\chi^\alpha)_1\sigma^\mu_{\alpha\dot{\beta}}(\chi^{\dagger\dot{\beta}})_2 \\ &= \chi_1 \cdot \sigma^\mu \cdot \chi_2^\dagger. \end{aligned}$$

For the second identity we use $\sigma^\mu = \epsilon(\bar{\sigma}^\mu)^T\epsilon^T$ which can be written as $\sigma^\mu_{\alpha\dot{\alpha}} = \epsilon_{\alpha\beta}(\bar{\sigma}^{\mu\dot{\beta}\beta})^T(\epsilon_{\dot{\alpha}\beta})^T$.
As $\chi_1 \cdot \sigma^\mu \cdot \chi_2^\dagger$ is a number, we find

$$\begin{aligned} \chi_1 \cdot \sigma^\mu \cdot \chi_2^\dagger &= \left(\chi_1 \cdot \sigma^\mu \cdot \chi_2^\dagger\right)^T = \left((\chi^\alpha)_1\sigma^\mu_{\alpha\dot{\alpha}}(\chi^{\dagger\dot{\alpha}})_2\right)^T \\ &= \left((\chi^\alpha)_1\epsilon_{\alpha\beta}(\bar{\sigma}^{\mu\dot{\beta}\beta})^T(\epsilon_{\dot{\alpha}\dot{\beta}})^T(\chi^{\dagger\dot{\alpha}})_2\right)^T \\ &= \left((\chi_\beta)_1(\bar{\sigma}^{\mu\dot{\beta}\beta})^T(\chi^\dagger_{\dot{\beta}})_2\right)^T \\ &= -(\chi^\dagger_{\dot{\beta}})_2\bar{\sigma}^{\mu\dot{\beta}\beta}(\chi_\beta)_1 = -\chi_2^\dagger \cdot \bar{\sigma}^\mu \cdot \chi_1 \qquad (1) \end{aligned}$$

g) The Lagrangian for a free particle can be written using chiral components:

$$\begin{aligned}
\mathcal{L}_{Dirac} &= \bar{\psi}(i\not{\partial} - m)\psi = (\chi^{\alpha}, \xi^{\dagger}_{\dot{\alpha}})[i\partial_{\mu}\gamma^{\mu} - m]\begin{pmatrix}\xi_{\alpha}\\ \chi^{\dagger\dot{\alpha}}\end{pmatrix} \\
&= (\chi^{\alpha}, \xi^{\dagger}_{\dot{\alpha}})\left[i\partial_{\mu}\begin{pmatrix}0 & \sigma^{\mu}_{\alpha\dot{\alpha}}\\ \bar{\sigma}^{\mu\dot{\alpha}\alpha} & 0\end{pmatrix} - m\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right]\begin{pmatrix}\xi_{\alpha}\\ \chi^{\dagger\dot{\alpha}}\end{pmatrix} \\
&= (\chi^{\alpha}, \xi^{\dagger}_{\dot{\alpha}})\left[i\partial_{\mu}\begin{pmatrix}\sigma^{\mu}_{\alpha\dot{\alpha}}\chi^{\dagger\dot{\alpha}}\\ \bar{\sigma}^{\mu\dot{\alpha}\alpha}\xi_{\alpha}\end{pmatrix} - m\begin{pmatrix}\xi_{\alpha}\\ \chi^{\dagger\dot{\alpha}}\end{pmatrix}\right] \\
&= \chi\cdot\sigma^{\mu}\cdot(i\partial_{\mu}\chi^{\dagger}) + \xi^{\dagger}\cdot\bar{\sigma}^{\mu}\cdot(i\partial_{\mu}\xi) - m\left(\chi\cdot\xi + \xi^{\dagger}\cdot\chi^{\dagger}\right).
\end{aligned}$$