Physics Beyond the Standard Model

Exercise Sheet 3 - Solutions

Exercise 1: Breaking of $SU(5)$

The Higgs vacuum expectation value (vev) is[1]

$$\phi_0 = \frac{v_{24}}{\sqrt{15}} diag\,(1,1,1,-3/2,-3/2)\,. \tag{1}$$

Then we can write the Higgs field as vev plus a fluctuation h:

$$\phi = \frac{v_{24}+h}{\sqrt{15}} diag\,(1,1,1,-3/2,-3/2)\,. \tag{2}$$

$$\begin{aligned}
Tr|\phi|^2 &= \left(\frac{v_{24}+h}{\sqrt{15}}\right)^2 Tr\,[diag(1,1,1,9/4,9/4)] = \frac{1}{2}(v_{24}+h)^2 \\
Tr|\phi|^4 &= \left(\frac{v_{24}+h}{\sqrt{15}}\right)^4 Tr\,[diag(1,1,1,81/16,81/16)] = \frac{7}{120}(v_{24}+h)^4 \\
\Longrightarrow V(\phi) &= \frac{1}{2}M_{24}^2(v_{24}+h)^2 + \frac{1}{4}\lambda_{24}(v_{24}+h)^4 + \frac{7}{120}\lambda'_{24}(v_{24}+h)^4
\end{aligned}$$

The potential has two critical points

$$\begin{aligned}
\left.\frac{dV(\phi)}{dh}\right|_{h=0} &= v_{24}M_{24}^2 + v_{24}^3\lambda_{24} + \frac{7}{30}v_{24}\lambda'_{24} \stackrel{!}{=} 0 \\
\Rightarrow v_{24}^2 &= -\frac{M_{24}^2}{\lambda_{24}+\frac{7}{30}\lambda'_{24}} \quad \text{oder} \quad v_{24}=0.
\end{aligned}$$

The condition for a minimum of the potential is

$$\left.\frac{d^2V(\phi)}{dh^2}\right|_{h=0} = M_{24}^2 + 3\lambda_{24}v_{24}^2 + \frac{7}{10}\lambda'_{24}v_{24}^2 \stackrel{!}{>} 0. \tag{3}$$

[1] For a detailed derviation of the vev of spontaneously broken gauge symmetries see L. F. Li, Phys. Rev. D9, 1723 (1974).

The conditions for a convex potential with non-trivial minimum are $M_{24}^2 < 0$ und $\lambda'_{24} > 0$.
Furthermore, v_{24} needs to be real $\Rightarrow \lambda_{24} + \frac{7}{30}\lambda'_{24} > 0$

$$\Longrightarrow \boxed{\lambda_{24} > -\frac{7}{30}\lambda'_{24}}$$

V_μ is a linear combination of the generators of $SU(5)$:
$V_\mu \equiv V_\mu^a T^a \qquad a = 1, \ldots, 24$

8 for $SU(3)$ (Gluons)
4 for $SU(2) \times U(1)$ $(W^\pm,\ Z,\ \gamma)$
6 Leptoquarks $(X_1,\ X_2,\ X_3,\ X_1^\dagger,\ X_2^\dagger,\ X_3^\dagger)$
6 Diquarks $(Y_1,\ Y_2,\ Y_3,\ Y_1^\dagger,\ Y_2^\dagger,\ Y_3^\dagger)$

We can calculate the masses from the Higgs vev ϕ_0:

$$\begin{aligned} Tr|D_\mu\phi_0|^2 &= Tr|\partial_\mu\phi_0 + ig_5[V_\mu, \phi_0]|^2 = Tr|ig_5[V_\mu, \phi_0]|^2 \\ &= g_5^2 Tr|[V_\mu, \phi_0]|^2 = \sum_i \left(M_{X_i}^2 X_i^\dagger X_i + M_{Y_i}^2 Y_i^\dagger Y_i \right) \end{aligned}$$

The vector potential is $V_\mu = \frac{1}{\sqrt{2}} \begin{pmatrix} A & B \\ B^\dagger & C \end{pmatrix}$, where A is a 3x3 matrix and C is a 2x2 matrix. A and C contain the $SU(3)$ and $SU(2) \times U(1)$ gauge fields. B is a 3x2 matrix and contains the leptoquark und diquark fields X, Y.
The Higgs vev is $\phi_0 = \frac{v_{24}}{\sqrt{15}} \begin{pmatrix} \mathbb{1}_{3x3} & 0 \\ 0 & -3/2 \cdot \mathbb{1}_{2x2} \end{pmatrix}$

$$\begin{aligned} [V_\mu, \phi_0] &= \frac{v_{24}}{\sqrt{30}} \left[\begin{pmatrix} A & B \\ B^\dagger & C \end{pmatrix} \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -3/2 \cdot \mathbb{1} \end{pmatrix} - \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -3/2 \cdot \mathbb{1} \end{pmatrix} \begin{pmatrix} A & B \\ B^\dagger & C \end{pmatrix} \right] \\ &= \frac{v_{24}}{\sqrt{30}} \begin{pmatrix} 0 & -5/2 B \cdot \mathbb{1} \\ 5/2 B^\dagger \cdot \mathbb{1} & 0 \end{pmatrix} \end{aligned}$$

$$
\begin{aligned}
\Rightarrow g_5^2 Tr|[V_\mu,\phi_0]|^2 &= \frac{25}{4\cdot 30} g_5^2 v_{24}^2 Tr\begin{pmatrix} BB^\dagger\cdot\mathbb{1} & 0 \\ 0 & BB^\dagger\cdot\mathbb{1}\end{pmatrix} \\
BB^\dagger &= \begin{pmatrix} X_1^\dagger & Y_1^\dagger \\ X_2^\dagger & Y_2^\dagger \\ X_3^\dagger & Y_3^\dagger\end{pmatrix}\begin{pmatrix} X_1 & X_2 & X_3 \\ Y_1 & Y_2 & Y_3\end{pmatrix} \\
&= \begin{pmatrix} X_1^\dagger X_1 + Y_1^\dagger Y_1 & \dots & \dots \\ \dots & X_2^\dagger X_2 + Y_2^\dagger Y_2 & \dots \\ \dots & \dots & X_3^\dagger X_3 + Y_3^\dagger Y_3\end{pmatrix} \\
g_5^2 Tr|[V_\mu,\phi_0]|^2 &= \frac{5}{12} g_5^2 v_{24}^2 \Big[X_1^\dagger X_1 + Y_1^\dagger Y_1 + X_2^\dagger X_2 + Y_2^\dagger Y_2 + X_3^\dagger X_3 + Y_3^\dagger Y_3\Big]
\end{aligned}
$$

$$
\boxed{M_X^2 \;=\; M_Y^2 \;=\; \frac{5}{12} g_5^2 v_{24}^2}
$$

Exercise 2: Running coupling constants and unification

Def. GUT scale: $\alpha_0 \equiv \alpha_i(M_{GUT}),\ \forall i$

Renormalization $\Longrightarrow$ running coupling constants:

$$\text{QED: } \alpha(\mu) \;=\; \frac{e^2(\mu)}{4\pi} \;=\; \frac{g^2(\mu)\sin^2\theta_W(\mu)}{4\pi} \;=\; \frac{g'^2(\mu)\cos^2\theta_W(\mu)}{4\pi}$$

$$\begin{aligned} \Longrightarrow \text{Hypercharge } Y\text{:} \quad & \alpha_1(\mu) \;=\; \frac{g'^2(\mu)}{4\pi}\cot^2\theta_W^{GUT} \qquad \text{mit } \cot\theta_W^{GUT} \;=\; \sqrt{\frac{5}{3}} \\ \text{Isospin } I_3\text{:} \quad & \alpha_2(\mu) \;=\; \frac{g^2(\mu)}{4\pi} \\ \text{QCD:} \quad & \alpha_3(\mu) \;=\; \frac{g_s^2(\mu)}{4\pi} \end{aligned}$$

$\alpha(\mu)$ as function of $\alpha_1(\mu)$, $\alpha_2(\mu)$:

$$\begin{aligned} \sin^2\theta_W(\mu) \;&=\; \frac{g'^2(\mu)}{g'^2(\mu)+g^2(\mu)} \;=\; \frac{\alpha_1(\mu)\tan^2\theta_W^{GUT}}{\alpha_1(\mu)\tan^2\theta_W^{GUT}+\alpha_2(\mu)} \;=\; \frac{3\alpha_1(\mu)}{3\alpha_1(\mu)+5\alpha_2(\mu)} \\ \frac{1}{\alpha(\mu)} \;&=\; \frac{1}{\alpha_2(\mu)\sin^2\theta_W(\mu)} \;=\; \frac{1}{\alpha_2(\mu)} + \frac{5}{3}\frac{1}{\alpha_1(\mu)} \end{aligned} \tag{1}$$

Renormalzation group equations (RGEs) (5) with $\alpha_i^Z \;\equiv\; \alpha_i(M_Z)$:

$$\frac{1}{\alpha_1^Z} \;=\; \frac{1}{\alpha_0} + \frac{1}{4\pi}\log\left(\frac{M_Z^2}{M_{GUT}^2}\right)\left(-\frac{4}{3}N_F - \frac{1}{10}N_H\right) \tag{2}$$

$$\frac{1}{\alpha_2^Z} \;=\; \frac{1}{\alpha_0} + \frac{1}{4\pi}\log\left(\frac{M_Z^2}{M_{GUT}^2}\right)\left(\underbrace{\frac{22}{3}}_{(*)} - \frac{4}{3}N_F - \frac{1}{6}N_H\right) \tag{3}$$

$$\frac{1}{\alpha_3^Z} \;=\; \frac{1}{\alpha_0} + \frac{1}{4\pi}\log\left(\frac{M_Z^2}{M_{GUT}^2}\right)\left(\underbrace{11}_{(**)} - \frac{4}{3}N_F\right) \tag{4}$$

$$\begin{aligned} (*) \;&=\; \text{non-abelian terms: self interaction of the gauge bosons} \\ (**) \;&=\; \text{asymptotic freedom of QCD} \end{aligned}$$

We can calculate the GUT scale from (1) and (4):

$$\begin{aligned}
\frac{1}{\alpha^Z} &= \frac{1}{\alpha_2^Z} + \frac{5}{3}\frac{1}{\alpha_1^Z} \\
&= \frac{1}{\alpha_0} + \frac{1}{4\pi}\log\left(\frac{M_Z^2}{M_{GUT}^2}\right)\left[\frac{22}{3} - \frac{4}{3}N_F - \frac{1}{6}N_H\right] \\
&+ \frac{5}{3}\left\{\frac{1}{\alpha_0} + \frac{1}{4\pi}\log\left(\frac{M_Z^2}{M_{GUT}^2}\right)\left[-\frac{4}{3}N_F - \frac{1}{10}N_H\right]\right\} \\
&= \frac{8}{3\alpha_0} + \frac{1}{4\pi}\log\left(\frac{M_Z^2}{M_{GUT}^2}\right)\left[\frac{22}{3} - \frac{32}{9}N_F - \frac{1}{3}N_H\right] \\
\frac{1}{\alpha_3^Z} &= \frac{1}{\alpha_0} + \frac{1}{4\pi}\log\left(\frac{M_Z^2}{M_{GUT}^2}\right)\left(11 - \frac{4}{3}N_F\right)
\end{aligned}$$

$$\begin{aligned}
\Longrightarrow \frac{1}{\alpha^Z} - \frac{8}{3\alpha_3^Z} &= \frac{1}{4\pi}\log\left(\frac{M_Z^2}{M_{GUT}^2}\right)\left[\frac{22}{3} - \frac{32}{9}N_F - \frac{1}{3}N_H - \frac{8}{3}\left(11 - \frac{4}{3}N_F\right)\right] \\
&= \frac{1}{4\pi}\log\left(\frac{M_{GUT}^2}{M_Z^2}\right)\left[22 + \frac{1}{3}N_H\right] \\
\Longrightarrow 2\log\left(\frac{M_{GUT}}{M_Z}\right) &= \frac{4\pi}{22 + \frac{N_H}{3}}\left(\frac{1}{\alpha^Z} - \frac{8}{3\alpha_3^Z}\right)
\end{aligned}$$

$$\boxed{\log\left(\frac{M_{GUT}}{\mu}\right) = \frac{2\pi}{11(1 + \frac{N_H}{66})}\left(\frac{1}{2\alpha(\mu)} - \frac{4}{3\alpha_3(\mu)}\right)}$$

Numerically: $M_{GUT} = 5.64 \cdot 10^{14} \ldots\ 8.62 \cdot 10^{14}$

We can calculate the Weinberg angle at the scale M_Z from (1), (2) and (4):

$$\begin{aligned}
\frac{1}{\alpha_2^Z} &= \frac{1}{\alpha_0} + \frac{1}{4\pi}\log\left(\frac{M_{GUT}^2}{M_Z^2}\right)\left(-\frac{22}{3}+\frac{4}{3}N_F+\frac{1}{6}N_H\right) \\
&= \frac{1}{\alpha_3^Z} + \frac{1}{4\pi}2\log\left(\frac{M_{GUT}}{M_Z}\right)\left[11-\frac{4}{3}N_F-\frac{22}{3}+\frac{4}{3}N_F+\frac{1}{6}N_H\right] \\
&= \frac{1}{\alpha_3^Z} + 2\cdot\frac{1}{4\pi}\left[\frac{2\pi}{11(1+\frac{N_H}{66})}\right]\left(\frac{1}{2\alpha(\mu)}-\frac{4}{3\alpha_3(\mu)}\right)\left[\frac{11}{3}+\frac{1}{6}N_H\right] \\
&= \frac{1}{\alpha_3^Z}\left[1+\frac{\frac{11}{3}+\frac{N_H}{6}}{11(1+\frac{N_H}{66})}\left(-\frac{4}{3}\right)\right] + \frac{1}{\alpha^Z}\left[\frac{1}{2}\frac{\frac{11}{3}+\frac{N_H}{6}}{11(1+\frac{N_H}{66})}\right] \\
&= \frac{1}{\alpha_3^Z}\left(\frac{\frac{5}{9}-\frac{N_H}{198}}{1+\frac{N_H}{66}}\right) + \frac{1}{\alpha^Z}\left(\frac{\frac{1}{6}+\frac{N_H}{132}}{1+\frac{N_H}{66}}\right) \\
&= \frac{5}{9\alpha_3^Z}\left(\frac{1-\frac{N_H}{110}}{1+\frac{N_H}{66}}\right) + \frac{1}{6\alpha^Z}\left(\frac{1+\frac{N_H}{22}}{1+\frac{N_H}{66}}\right)
\end{aligned}$$

$$\boxed{\sin^2\theta_W(M_Z) = \frac{\alpha(M_Z)}{\alpha_2(M_Z)} = \frac{1}{6}\left(\frac{1+\frac{N_H}{22}}{1+\frac{N_H}{66}}\right) + \frac{5}{9}\left(\frac{1-\frac{N_H}{110}}{1+\frac{N_H}{66}}\right)\frac{\alpha(M_Z)}{\alpha_3(M_Z)}}$$

Numerically: $\sin^2\theta_W(M_Z) = 0.209\ \ldots\ 0.206$