Physics Beyond The Standard Model

Exercise sheet 2 - Solutions

Exercise 1: QED Lagrangian

$\mathcal{L} \longrightarrow$ Equations of motion from Euler-Lagrange equation:

$$\boxed{\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} - \frac{\partial \mathcal{L}}{\partial \phi} = 0}$$

$$\begin{aligned}
1) \qquad \mathcal{L} &= |D_\mu \phi|^2 - m^2 |\phi|^2 \\
&= \Big[\Big(\partial_\mu + ieA_\mu Q\Big)\phi\Big]^\dagger \Big[\Big(\partial_\mu + ieA_\mu Q\Big)\phi\Big] - m^2 \phi^\dagger \phi \\
&\quad \text{with charge operator eigenvalue } \mathcal{Q}\phi \;=\; Q\phi \\
\frac{\partial \mathcal{L}}{\partial \phi^*} &= \Big(-ieA_\mu Q\Big)\Big[\Big(\partial_\mu + ieA_\mu Q\Big)\phi\Big] - m^2 \phi \\
\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^*)} &= \partial_\mu \Big[\Big(\partial_\mu + ieA_\mu Q\Big)\phi\Big] \\
&= \partial_\mu \partial_\mu \phi + ie(\partial_\mu A_\mu) Q\phi + ieA_\mu Q \partial_\mu \phi
\end{aligned}$$

Euler-Lagrange equation:

$$\partial_\mu \partial_\mu \phi + ie(\partial_\mu A_\mu) Q\phi + ieA_\mu Q\partial_\mu \phi + ieA_\mu Q\partial_\mu \phi - e^2 A_\mu A_\mu Q^2 \phi + m^2 \phi = 0$$

$$\begin{aligned}
\Rightarrow \partial_\mu \partial_\mu \phi + m^2 \phi &= -ieA_\mu Q\partial_\mu \phi + e^2 A_\mu A_\mu Q^2 \phi - ie(\partial_\mu A_\mu) Q\phi - ieA_\mu Q\partial_\mu \phi \\
&= -ieQ\partial_\mu (A_\mu \phi) - eQA_\mu (i\partial_\mu - eA_\mu Q)\phi
\end{aligned}$$

$$\Longrightarrow \qquad \boxed{\Big[\partial^2 + m^2\Big]\phi = -ieQ\partial_\mu (A_\mu \phi) - ieQA_\mu D_\mu \phi}$$

$D_\mu \Rightarrow \partial_\mu : \Big[\partial^2 + m^2\Big]\phi \;=\; 0 \quad \stackrel{\triangle}{=} \quad$ Klein-Gordon equation

2) $\mathcal{L} = \bar{\psi}\Big(i \not{D} - m\Big)\psi$ with $\bar{\psi} = \psi^\dagger\gamma_0$ and $\not{D} = \gamma_\mu D_\mu$

$= \bar{\psi}\Big[i(\not{\partial} + ie \not{A}Q) - m\Big]\psi = \bar{\psi}\Big[i \not{\partial} - e \not{A}Q - m\Big]\psi$

$$\frac{\delta\mathcal{L}}{\delta\bar{\psi}} = \Big[i\gamma_\mu\partial_\mu - e \not{A}Q - m\Big]\psi \qquad \& \qquad \partial_\mu\left(\frac{\delta\mathcal{L}}{\delta(\partial_\mu\bar{\psi})}\right) = 0$$

Euler-Lagrange equation:

$$\Big[i \not{\partial} - e \not{A}Q - m\Big]\psi = 0 \qquad \Longrightarrow \qquad \boxed{\Big(i \not{\partial} - m\Big)\psi = e \not{A}Q\psi}$$

$D_\mu \Rightarrow \partial_\mu : \Big(i \not{\partial} - m\Big)\psi = 0 \quad \stackrel{\wedge}{=} \quad$ Dirac equation

$$\frac{\delta\mathcal{L}}{\delta\psi} = \bar{\psi}\Big[- e \not{A}Q - m\Big] \qquad \& \qquad \partial_\mu\left(\frac{\delta\mathcal{L}}{\delta(\partial_\mu\psi)}\right) = \partial_\mu\bar{\psi}\Big[i\gamma_\mu\Big]$$

Euler-Lagrange equation:

$$\Big[i\partial_\mu\bar{\psi}\gamma_\mu + e\bar{\psi} \not{A}Q + m\bar{\psi}\Big] = 0 \quad \Longrightarrow \quad \boxed{\bar{\psi}(i \overleftarrow{\not{\partial}} + m) = -e\bar{\psi} \not{A}Q}$$

3)
$$\begin{aligned}
\mathcal{L} &= -\frac{1}{4}F_{\mu\nu}F_{\mu\nu} - j_\mu A_\mu \qquad \text{with } F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu \\
&= -\frac{1}{4}\Big(\partial_\mu A_\nu - \partial_\nu A_\mu\Big)\Big(\partial_\mu A_\nu - \partial_\nu A_\mu\Big) - j_\mu A_\mu \\
&= -\frac{1}{4}\Big[\partial_\mu A_\nu \partial_\mu A_\nu - \partial_\mu A_\nu \partial_\nu A_\mu - \partial_\nu A_\mu \partial_\mu A_\nu + \partial_\nu A_\mu \partial_\nu A_\mu\Big] - j_\mu A_\mu \\
&= -\frac{1}{2}\Big[(\partial_\alpha A_\beta)^2 - (\partial_\alpha A_\beta)(\partial_\beta A_\alpha)\Big] - j_\alpha A_\alpha
\end{aligned}$$

$$\begin{aligned}
\frac{\delta\mathcal{L}}{\delta A_\mu} &= -j_\mu \quad \& \quad \partial_\mu\left(\frac{\delta\mathcal{L}}{\delta(\partial_\mu A_\nu)}\right) = -\frac{1}{2}\partial_\mu\Big(2\partial_\mu A_\nu - \partial_\nu A_\mu - \partial_\nu A_\mu\Big) \\
&\qquad\qquad = -\partial_\mu\Big[\partial_\mu A_\nu - \partial_\nu A_\mu\Big] \\
&\qquad\qquad = -\partial_\mu F_{\mu\nu}
\end{aligned}$$

Euler-Lagrange equation:

$$-\partial_\mu F_{\mu\nu} + j_\nu = 0 \qquad \Longrightarrow \qquad \boxed{\partial_\mu F_{\mu\nu} = j_\nu} \qquad \text{Maxwell's equation}$$

Aufgabe 2: Gauge boson masses

The standard model lagrangian needs to be $SU(2)_L \times U(1)_Y$ invariant. This means that explicit mass terms cannot be introduced because they break the gauge invariance:

$$\begin{aligned}
\mathcal{L}_m &= -m\bar{\psi}\psi \\
&= -m(\bar{\psi}_L + \bar{\psi}_R)(\psi_L + \psi_R) = -m[\bar{\psi}_L\psi_L + \bar{\psi}_L\psi_R + \bar{\psi}_R\psi_L + \bar{\psi}_R\psi_R] \\
&= -m(\psi^\dagger P_L\gamma^0 + \psi^\dagger P_R\gamma^0)(P_L\psi + P_R\psi) \\
&= -m[\psi^\dagger\gamma^0 P_R + \psi^\dagger\gamma^0 P_L)(P_L\psi + P_R\psi] \\
&= -m[\bar{\psi}P_R P_L\psi + \bar{\psi}P_R P_R\psi + \bar{\psi}P_L P_L\psi + \bar{\psi}P_L P_R\psi] \\
&= -m[\bar{\psi}P_R P_R\psi + \bar{\psi}P_L P_L\psi] \\
&= -m[\bar{\psi}_L\psi_R + \bar{\psi}_R\psi_L]
\end{aligned}$$

These mixed terms are not invariant under $SU(2)$ because $I(\psi_R) = 0$ and $I(\psi_L) = \pm 1/2$.

The masses can be generated using the Higgs mechanism. The complete Lagrangian of the electroweak standard model is:

$$\mathcal{L} = -\frac{1}{4} W_{\mu\nu} \cdot W^{\mu\nu} - \frac{1}{4} B_{\mu\nu} B^{\mu\nu} \qquad \left\{ \begin{array}{l} W^{\pm}, Z, \gamma \text{ kinetic terms} \\ \text{and self interactions} \end{array} \right.$$

$$+ \overline{L} \gamma^{\mu} \left(i\partial_{\mu} - g \frac{\sigma_a}{2} \cdot W_{\mu}^{a} - g' \frac{Y}{2} B_{\mu} \right) L \qquad \left\{ \begin{array}{l} \text{Lepton kinetic terms and} \\ \text{Quarks kinetic terms and} \\ \text{their interactions with } W^{\pm}, Z, \gamma \end{array} \right.$$

$$+ \overline{R} \gamma^{\mu} \left(i\partial_{\mu} - g' \frac{Y}{2} B_{\mu} \right) R$$

$$+ \Big| \underbrace{\left(i\partial_{\mu} - g \frac{\sigma_a}{2} \cdot W_{\mu}^{a} - g' \frac{Y}{2} B_{\mu} \right) \phi}_{(*)} \Big|^2 - V(\phi) \qquad \left\{ \begin{array}{l} W^{\pm}, Z, \gamma \text{ and Higgs} \\ \text{masses and couplings} \end{array} \right.$$

$$-(G_1 \overline{L} \phi R + G_2 \overline{L} \phi^c R + h.c.) \qquad \left\{ \begin{array}{l} \text{Lepton and quark masses and} \\ \text{Yukawa couplings to the Higgs} \end{array} \right.$$

Spontaneous symmetry breaking: $\phi(x) = \phi_0 + \delta\phi(x)$ with $\phi_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix}$ with Isospin $I = 1/2$ and hyper charge $Y = 1$ for the Higgs field ϕ_0.

$$\Rightarrow (*) = \underbrace{\Big| \left(ig \frac{\sigma^a}{2} \cdot W_{\mu}^{a} + ig' \frac{Y}{2} B_{\mu} \right) \phi_0 \Big|^2}_{(**)} + \ \ldots \text{ other terms}$$

Pauli matrices: $\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ and $\sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$$
\begin{aligned}
(**) &= \frac{1}{8}\left| \begin{pmatrix} gW_\mu^3 + g'B_\mu & g(W_\mu^1 - iW_\mu^2) \\ g(W_\mu^1 + iW_\mu^2) & -gW_\mu^3 + g'B_\mu \end{pmatrix} \begin{pmatrix} 0 \\ v \end{pmatrix} \right|^2 \\
&= \frac{v^2}{8}\left\{ g^2 |W_\mu^1 - iW_\mu^2|^2 + (-gW_\mu^3 + g'B_\mu)^2 \right\} \\
&= \frac{v^2}{8}\Bigg\{ \underbrace{g^2\Big[(W_\mu^1)^2 + (W_\mu^2)^2\Big]}_{(1)} + \underbrace{\Big[(-gW_\mu^3 + g'B_\mu)(-gW_\mu^3 + g'B_\mu)\Big]}_{(2)} \Bigg\} \\
&= \frac{v^2}{8} \begin{pmatrix} W_\mu^1 \\ W_\mu^2 \\ W_\mu^3 \\ B_\mu \end{pmatrix} \begin{pmatrix} g^2 & 0 & 0 & 0 \\ 0 & g^2 & 0 & 0 \\ 0 & 0 & g^2 & -gg' \\ 0 & 0 & -gg' & g'^2 \end{pmatrix} \begin{pmatrix} W_\mu^1 \\ W_\mu^2 \\ W_\mu^3 \\ B_\mu \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
(1) &= \frac{1}{8}(vg)^2 \left(W_\mu^1 - iW_\mu^2\right)\left(W_\mu^1 + iW_\mu^2\right) \\
&= \left(\frac{1}{2}vg\right)^2 W_\mu^+ W_\mu^- \qquad\qquad \text{with} \qquad W_\mu^\pm \equiv \frac{1}{\sqrt{2}}\left(W_\mu^1 \mp iW_\mu^2\right) \\
\Longrightarrow &\quad \boxed{M_{W^\pm} = \frac{1}{2}vg} \text{ from the eigenvalues of } \frac{1}{4}v^2 \begin{pmatrix} g^2 & 0 \\ 0 & g^2 \end{pmatrix}
\end{aligned}
$$

(2) we want to find the physical states Z_μ, A_μ and their masses:

$$
\begin{aligned}
\begin{pmatrix} Z_\mu \\ A_\mu \end{pmatrix} &= \begin{pmatrix} \cos\theta_W & -\sin\theta_W \\ \\ \sin\theta_W & \cos\theta_W \end{pmatrix} \begin{pmatrix} W_\mu^3 \\ B_\mu \end{pmatrix} \equiv I\!R \begin{pmatrix} W_\mu^3 \\ B_\mu \end{pmatrix} \\
I\!M^2 &= I\!R \frac{v^2}{4} \begin{pmatrix} g^2 & -gg' \\ -gg' & g'^2 \end{pmatrix} I\!R^T \\
&= \frac{v^2}{4} \begin{pmatrix} g^2 c_W^2 + 2gg' s_W c_W + g'^2 s_W^2 & \left(g^2 - g'^2\right) s_W c_W + gg'(s_W^2 - c_W^2) \\ \left(g^2 - g'^2\right) s_W c_W + gg'(s_W^2 - c_W^2) & g^2 s_W^2 - 2gg' s_W c_W + g'^2 c_W^2 \end{pmatrix} \\
&\stackrel{!}{=} \frac{v^2}{4} \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}
\end{aligned}
$$

off diagonal elements vanish $\Leftrightarrow \dfrac{g'}{g} = \tan\theta_W$

(or $\dfrac{g'}{g} = -\cot\theta_W$, which yields the same with A and B exchanged)

$$\left.\begin{array}{l} A = \left(\dfrac{g'}{s_W}\right)^2 = g^2 + g'^2 \\ B = 0 \end{array}\right\} \quad \Longrightarrow \quad \mathbb{M}^2 = \frac{v^2}{4}\begin{pmatrix} g^2+g'^2 & 0 \\ 0 & 0 \end{pmatrix}$$

The eigenvalues of $\mathbb{M}^2$ are $\dfrac{v^2}{4}\left(g^2+g'^2\right)$ and 0.
Calculation of the eigenvectors

$$\begin{aligned} \begin{pmatrix} g^2 & -gg' \\ -gg' & g'^2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= \frac{v^2}{4}\left(g^2+g'^2\right)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \Rightarrow \quad x_1 = -\frac{g}{g'}x_2 \\ \begin{pmatrix} g^2 & -gg' \\ -gg' & g'^2 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} &= 0\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \quad \Rightarrow \quad y_1 = \frac{g'}{g}y_2 \end{aligned}$$

where the normalisation N is not fixed yet. Base transformation from $\begin{pmatrix} W_\mu^3 \\ B_\mu \end{pmatrix}$ to $\begin{pmatrix} Z_\mu \\ A_\mu \end{pmatrix}$:

$$\begin{pmatrix} Z_\mu \\ A_\mu \end{pmatrix} = \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix}\begin{pmatrix} W_\mu^3 \\ B_\mu \end{pmatrix} = \underbrace{N\begin{pmatrix} g & -g' \\ g' & g \end{pmatrix}}_{IR}\begin{pmatrix} W_\mu^3 \\ B_\mu \end{pmatrix} = N\begin{pmatrix} gW_\mu^3 - g'B_\mu \\ g'W_\mu^3 + gB_\mu \end{pmatrix}$$

$$IRIR^T = 1 \Longrightarrow N = \frac{1}{\sqrt{g^2+g'^2}}$$

so

$$\begin{aligned} \frac{v^2}{8}\left(-gW_\mu^3 + g'B_\mu\right)^2 &= \frac{1}{8}v^2\left[\left(g^2+g'^2\right)N^2\left(gW_\mu^3 - g'B_\mu\right)^2 + 0N^2\left(g'W_\mu^3 + gB_\mu\right)^2\right] \\ &= \frac{1}{8}v^2\left[\left(g^2+g'^2\right)\frac{\left(gW_\mu^3 - g'B_\mu\right)^2}{g^2+g'^2} + 0\frac{\left(g'W_\mu^3 + gB_\mu\right)^2}{g^2+g'^2}\right] \\ &= \frac{1}{2}M_Z^2 Z_\mu^2 + \frac{1}{2}M_A^2 A_\mu^2 \end{aligned}$$

$$\Longrightarrow \qquad \boxed{\begin{array}{lll} Z_\mu = \dfrac{gW_\mu^3 - g'B_\mu}{\sqrt{g^2+g'^2}} & \& & M_Z \;=\; \dfrac{v}{2}\sqrt{g^2+g'^2} \\ \\ A_\mu = \dfrac{g'W_\mu^3 + gB_\mu}{\sqrt{g^2+g'^2}} & \& & M_A \;=\; 0 \end{array}}$$