Physics Beyond The Standard Model

# Exercise sheet 1 - Solutions

**Exercise 1:** $\underline{e^+e^- \to \mu^+\mu^-}$

total cross-section

$$\begin{aligned}
\sigma_{tot} &= \int_{-1}^{1} \frac{d\sigma}{d\cos\theta} d\cos\theta \\
&= \frac{\pi\alpha^2}{2s}\beta \int_{-1}^{1} \Big\{ Q_1 \left(1+\beta^2\cos^2\theta\right) + Q_2\left(1-\beta^2\right) + Q_3 2\beta\cos\theta \Big\} d\cos\theta \\
&= \frac{\pi\alpha^2}{2s}\beta \left[ Q_1\left(x+\beta^2\frac{x^3}{3}\right) + Q_2\left(1-\beta^2\right)x + Q_3\beta x^2 \right]_{-1}^{1}
\end{aligned}$$

$$\Longrightarrow \quad \boxed{\sigma_{tot} = \frac{\pi\alpha^2}{s}\beta\Big\{ Q_1\Big(1+\frac{1}{3}\beta^2\Big) + Q_2\Big(1-\beta^2\Big)\Big\}}$$

asymmetry:

$$\begin{aligned}
A_{FB} &= \frac{1}{\sigma_{tot}}\Big\{ \underbrace{\int_0^1 dx\ \frac{d\sigma}{dx}}_{(1)} - \underbrace{\int_{-1}^0 dx\ \frac{d\sigma}{dx}}_{(2)} \Big\} \\
(1) &= \frac{\pi\alpha^2}{2s}\beta \left[ Q_1\left(x+\beta^2\frac{x^3}{3}\right) + Q_2\left(1-\beta^2\right)x + Q_3\beta x^2 \right]_0^1 \\
&= \frac{\pi\alpha^2}{2s}\beta\Big\{ Q_1\left(1+\frac{\beta^2}{3}\right) + Q_2\left(1-\beta^2\right) + Q_3\beta \Big\} \\
(2) &= \frac{\pi\alpha^2}{2s}\beta\Big\{ Q_1\left(1+\frac{\beta^2}{3}\right) + Q_2\left(1-\beta^2\right) - Q_3\beta \Big\}
\end{aligned}$$

$$\Longrightarrow \quad \boxed{A_{FB} = \frac{Q_3\beta}{Q_1\left(1+\frac{1}{3}\beta^2\right) + Q_2\left(1-\beta^2\right)}}$$

How did we arrive at the expression for the differential cross-section?

Example: $e^-(k_1)e^+(k_2) \to \gamma(q) \to \mu^-(p_1)\mu^+(p_2)$ with $m_e = m_\mu = 0$

$$\begin{aligned}
\text{Amplitude: } iT &= \underbrace{\bar{u}_s(p_1)(-ie\gamma^\mu)v_s(p_2)}_{\mu-\text{part}} \underbrace{\left[-i\frac{g_{\mu\nu}}{q^2}\right]}_{\gamma-\text{Prop.}} \underbrace{\bar{v}_s(k_2)(-ie\gamma^\nu)u_s(k_1)}_{e-\text{part}} \\
&= i\frac{e^2}{s}\Big[\bar{u}_s(p_1)\gamma^\mu v_s(p_2)\Big]\Big[\bar{v}_s(k_2)\gamma_\mu u_s(k_1)\Big]
\end{aligned}$$

average squared matrix element:

$$\begin{aligned}
\overline{|\mathcal{M}|^2} &= \frac{1}{4}\sum_{Spin}|iT|^2 \\
&= \frac{e^4}{4s^2}\sum_s \Big[\bar{u}_s(p_1)\gamma^\mu v_s(p_2)\Big]\Big[\bar{v}_s(k_2)\gamma_\mu u_s(k_1)\Big]\cdot\left(\Big[\bar{u}_s(p_1)\gamma^\nu v_s(p_2)\Big]\Big[\bar{v}_s(k_2)\gamma_\nu u_s(k_1)\Big]\right)^\dagger \\
&\overset{(1)}{=} \frac{e^4}{4s^2}\sum_s \Big[\bar{u}_s(p_1)\gamma^\mu v_s(p_2)\Big]\Big[\bar{v}_s(k_2)\gamma_\mu u_s(k_1)\Big]\cdot\Big[\bar{u}_s(k_1)\gamma_\nu v_s(k_2)\Big]\Big[\bar{v}_s(p_2)\gamma^\nu u_s(p_1)\Big] \\
&= \frac{e^4}{4s^2}\sum_s \underbrace{\Big[\bar{v}_s(p_2)\gamma^\nu u_s(p_1)\Big]\Big[\bar{u}_s(p_1)\gamma^\mu v_s(p_2)\Big]}_{\mu-\text{part}}\underbrace{\Big[\bar{v}_s(k_2)\gamma_\mu u_s(k_1)\Big]\Big[\bar{u}_s(k_1)\gamma_\nu v_s(k_2)\Big]}_{e-\text{part}} \\
&= \frac{e^4}{4s^2}\sum_s Tr\Big[v_s(p_2)\bar{v}_s(p_2)\gamma^\nu u_s(p_1)\bar{u}_s(p_1)\gamma^\mu\Big]Tr\Big[v_s(k_2)\bar{v}_s(k_2)\gamma_\mu u_s(k_1)\bar{u}_s(k_1)\gamma_\nu\Big] \\
&\overset{(2)}{=} \frac{e^4}{4s^2}Tr\Big[(\not{p}_2 - m_\mu)\gamma^\nu(\not{p}_1 + m_\mu)\gamma^\mu\Big]\cdot Tr\Big[(\not{k}_2 - m_e)\gamma_\mu(\not{k}_1 + m_e)\gamma_\nu\Big] \\
&= \frac{e^4}{4s^2}Tr\Big[\ \not{p}_2\gamma^\nu\ \not{p}_1\gamma^\mu\Big]\cdot Tr\Big[\ \not{k}_2\gamma_\mu\ \not{k}_1\gamma_\nu\Big] \\
&\overset{(3)}{=} \frac{e^4}{4s^2}4\Big[p_2^\nu p_1^\mu + p_1^\nu p_2^\mu - g^{\nu\mu}(p_2p_1)\Big]\cdot 4\Big[k_{2\mu}k_{1\nu} + k_{1\mu}k_{2\nu} - g_{\mu\nu}(k_2k_1)\Big] \\
&= \frac{4e^4}{s^2}\Big[2(p_2k_2)(p_1k_1) + 2(p_2k_1)(p_1k_2) - 4(p_2p_1)(k_2k_1) + g^{\nu\mu}g_{\mu\nu}(p_2p_1)(k_2k_1)\Big] \\
&= \frac{8e^4}{s^2}\big[(p_2k_2)(p_1k_1) + (p_2k_1)(p_1k_2)\big]
\end{aligned}$$

with (1) $\gamma^0\gamma^{\mu\dagger}\gamma^0 = \gamma^\mu$

$$(2) \quad \sum_s u_s(k)\bar{u}_s(k) = (\not{k} + m) \quad \& \quad \sum_s v_s(p)\bar{v}_s(p) = (\not{p} - m)$$

and $$(3) \quad Tr(\not{p}\gamma_\mu \not{k}\gamma_\nu) = 4\Big[k_\mu p_\nu + p_\mu k_\nu - g_{\mu\nu}(p \cdot k)\Big]$$

$$\begin{aligned} \text{in CM system:} \quad k_1 &= \frac{\sqrt{s}}{2}(1,0,0,1) \quad \& \quad k_2 = \frac{\sqrt{s}}{2}(1,0,0,-1) \\ p_1 &= \frac{\sqrt{s}}{2}(1, \sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) \\ p_2 &= \frac{\sqrt{s}}{2}(1, -\sin\theta\cos\phi, -\sin\theta\sin\phi, -\cos\theta) \end{aligned}$$

so that

$$\begin{aligned} \overline{|\mathcal{M}|^2} &= \frac{8e^4}{s^2}\left[\frac{s^2}{16}\Big((1+\cos\theta)^2 + (1-\cos\theta)^2\Big)\right] \\ &= e^4\Big(1+\cos^2\theta\Big) = 16\pi^2\alpha^2\Big(1+\cos^2\theta\Big) \\ d\sigma &= \frac{\overline{|\mathcal{M}|^2}}{F} dPS_2 \quad \overset{CM}{\Longrightarrow} \quad \frac{d\sigma}{d\Omega}\Big|_{CM} = \frac{1}{64\pi^2 s}\overline{|\mathcal{M}|^2} \\ \Rightarrow \frac{d\sigma}{d\cos\theta} &= 2\pi\frac{d\sigma}{d\Omega} = \frac{\pi\alpha^2}{2s}\Big(1+\cos^2\theta\Big) \quad = \text{1st term for massless particle} \end{aligned}$$

What's the physical meaning of $Q_{1,2,3}$ ?

- $Q_2 \overset{\wedge}{=}$ mass term $(\beta^2 = 1\big|_{m=0})$
  The mass term mixes left and right handed components of the spinor. For $m = 0$ the term proportional to $Q_2$ needs to vanish because $Q_2$ is the

only mixing term and handedness is conserved for massless particles:

$$\begin{aligned}
Q_{LL}Q^*_{RL} &\triangleq \mathcal{M}\Big(e^-_L e^+_R \to \mu^-_L \mu^+_R\Big)\mathcal{M}^*\Big(e^-_L e^+_R \to \mu^-_R \mu^+_L\Big) \\
&\propto \sum_s \Big[\ldots \bar{u}_L(p_1)\gamma^\nu v_R(p_2)\bar{v}_L(p_2)\gamma^\mu u_R(p_1)\ldots\Big] \\
&\propto \sum_s \Big[\ldots P_R v(p_2) v^\dagger(p_2) P_L \gamma_0 \ldots\Big] \\
&\propto \sum_s \Big[\ldots P_R v(p_2)\bar{v}(p_2) P_R \ldots\Big] \\
&\propto \Big[\ldots P_R(\not{p}_2 - m)P_R \ldots\Big]
\end{aligned}$$

$$\begin{aligned}
P_{L/R} &= \frac{1}{2}(1 \mp \gamma_5) \quad \& \quad \gamma_5^2 = 1 \\
P_R P_L &= \frac{1}{4}(1+\gamma_5)(1-\gamma_5) = \frac{1}{4}(1-\gamma_5^2) = 0 \\
P_R P_R &= \frac{1}{4}(1+\gamma_5)(1+\gamma_5) = \frac{1}{4}(1+2\gamma_5+\gamma_5^2) = \frac{1}{2}(1+\gamma_5) = P_R \\
P_L P_L &= P_L \\
\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu &= 2g_{\mu\nu} \qquad \& \qquad \gamma_\mu\gamma_5 + \gamma_5\gamma_\mu = 0
\end{aligned}$$

$$\begin{aligned}
P_R\gamma^\mu P_R &= \frac{1}{2}(1-\gamma_5)\gamma^\mu\frac{1}{2}(1-\gamma_5) = \gamma^\mu\frac{1}{2}(1+\gamma_5)\frac{1}{2}(1-\gamma_5) = \gamma^\mu P_L P_R = 0 \\
P_L m P_L &= m P_L
\end{aligned}$$

This interference therefore only appears for non-zero masses.

- $Q_3$ can only come from the $Z$-exchange. This can be seen from the calculation of the $\gamma$-exchange or when setting $Q_{ab} = e_\mu e_e$ (i.e. pure QED): $Q_{LL} = Q_{LR} = Q_{RL} = Q_{RR} = \tilde{Q} \implies Q_1 = Q_2 = \tilde{Q}^2 \quad \& \quad Q_3 = 0$ Therefore the $Q_3$ term contains the left-right asymmetry due to the structure of the $Z$-boson fermion coupling.

Coming back to the result for the total cross-section: The total cross-section is independent of the left-right asymmetry and for massless particles equals $\sigma = \dfrac{4\pi\alpha^2}{3s}Q_1$.

The asymmetry is only due to the $Z$-exchange, which is why the numerator of $A_{FB}$ is proportional to $Q_3$.

**Exercise 2:** 2 particle phase-space

$$dPS_2 = \frac{d^3p_1}{(2\pi)^3 2p_1^0}\ \frac{d^3p_2}{(2\pi)^3 2p_2^0}\ (2\pi)^4\delta^{(4)}(k_1+k_2-p_1-p_2)$$

with $$\frac{d^3p_2}{2p_2^0} \overset{(1)}{=} \frac{d^3p_2}{2p_2^0}\ dp_2^0\ \delta\Big(p_2^0 - \sqrt{\vec{p}_2^{\,2}+m_2^2}\Big) \overset{(2)}{=} d^4p_2\ \delta\Big(p_2^2-m_2^2\Big)\theta(p_2^0)$$

$$(1)\quad \int dp_2^0 \delta\Big(p_2^0 - \sqrt{\vec{p}_2^{\,2}+m_2^2}\Big) = 1$$

$$(2)\quad \delta\Big(p_2^2-m_2^2\Big)\theta(p_2^0) = \frac{1}{2|p_2^0|}\delta\Big(p_2^0 - \sqrt{\vec{p}_2^{\,2}+m_2^2}\Big)$$

and
$$\begin{aligned} d^3p_1 &= |\vec{p_1}|^2\ d|\vec{p_1}|\ d\Omega \quad , \qquad d\Omega = d\cos\theta\ d\phi \\ |\vec{p_1}|^2 &= (p_1^0)^2 - m_1^2 \quad \Rightarrow |\vec{p_1}|\ d|\vec{p_1}| = p_1^0\ dp_1^0 \end{aligned}$$

$$\begin{aligned} \Longrightarrow dPS_2 &= \frac{1}{(2\pi)^2}\underbrace{d^4p_2\ \delta^{(4)}(k_1+k_2-p_1-p_2)}_{\Downarrow}\delta(p_2^2-m_2^2)\frac{|\vec{p_1}|\ p_1^0\ dp_1^0}{2p_1^0}\underbrace{\theta(\sqrt{s}-p_1^0)}_{=1,\ p_1^0\le\sqrt{s}}d\Omega \\ &\qquad\qquad p_2 = k_1+k_2-p_1 \\ &= \frac{1}{8\pi^2}|\vec{p_1}|\ dp_1^0\ \underbrace{\delta\big[(k_1+k_2-p_1)^2-m_2^2\big]}_{=\,s-2\sqrt{s}p_1^0+m_1^2-m_2^2}\ d\Omega \\ &\qquad\qquad \text{im CMS: } k_1+k_2=(\sqrt{s},\vec{0}) \\ &\qquad\qquad\qquad \Downarrow \\ &\qquad\qquad p_1^0 = \frac{s+m_1^2-m_2^2}{2\sqrt{s}} \quad \Rightarrow |\vec{p_1}| = \frac{\sqrt{\lambda(s,m_1^2,m_2^2)}}{2\sqrt{s}} \end{aligned}$$

$$\overset{(2)}{=} \quad \frac{1}{8\pi^2}\frac{\sqrt{\lambda(s,m_1^2,m_2^2)}}{2\sqrt{s}}\,dp_1^0\,\frac{1}{2\sqrt{s}}\,\delta\Big(p_1^0-\frac{s+m_1^2-m_2^2}{2\sqrt{s}}\Big)\,d\Omega$$

where $\lambda(x,y,z)\equiv x^2+y^2+z^2-2xy-2xz-2yz$ (Kaellen function).

$$\Longrightarrow \quad \boxed{dPS_2=\frac{\sqrt{\lambda(s,m_1^2,m_2^2)}}{32\pi^2\;s}\,d\cos\theta\;d\phi}$$

isotropic scattering $\Longrightarrow \quad \dfrac{\sqrt{\lambda}}{8\pi s}$

- $m_1 = m_2 = m \quad \Rightarrow \quad \dfrac{\sqrt{\lambda(s,m^2,m^2)}}{s} = \sqrt{1-\dfrac{4m^2}{s}} = \beta$

  isotropic scattering $\Longrightarrow PS_2 = \dfrac{\beta}{8\pi}$

- $m_1 = 0 \quad \Rightarrow \quad \dfrac{\sqrt{\lambda(s,0,m^2)}}{s} = 1-\dfrac{m^2}{s}$

  isotropic scattering $\Longrightarrow PS_2 = \dfrac{1-m^2/s}{8\pi}$

- $m_1 = m_2 = 0 \quad \Rightarrow \quad \dfrac{\sqrt{\lambda(s,0,0)}}{s} = 1$

  isotropic scattering $\Longrightarrow PS_2 = \dfrac{1}{8\pi}$