Physics Beyond The Standard Model

# Exercise sheet 4

**Exercise 1:** Grassmann numbers

A Grassmann number $\theta$ is an anti-commuting object: $\{\theta, \theta\} = 0$.

- What does this imply for the series expansion of a function $\phi(\theta)$ ?
- What does this mean for the usual translation invariance of an integral

$$\int_{-\infty}^{+\infty} dx\phi(x) = \int_{-\infty}^{+\infty} dx\phi(x+c)$$

  when considering an integral over Grassmann numbers

$$\int d\theta, \int \theta d\theta, \frac{\partial}{\partial\theta}$$

  The normalisation is such that $\int_{-\infty}^{\infty} \theta d\theta = 1$.
- How do these rules change when considering a set of Grassmann numbers $\Theta = \{\theta_1, \theta_2, \theta_3, \dots\}$?

**Exercise 2:** Weyl spinors and Dirac bispinors

Consider the Clifford algebra in the chiral basis:

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \qquad \gamma_5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

where

$$\begin{aligned} \sigma^\mu &= \sigma^\mu_{\alpha\dot{\alpha}} \qquad (\alpha, \dot{\alpha} = 1, 2) \\ \bar{\sigma}^\mu &= \bar{\sigma}^{\mu\dot{\alpha}\alpha} \end{aligned}$$

with the Pauli matrices

$$\sigma^0 = \bar{\sigma}^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \sigma^1 = -\bar{\sigma}^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
$$\sigma^2 = -\bar{\sigma}^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma^3 = -\bar{\sigma}^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

A Dirac bispinor and its chiral components are defined as

$$\psi = \begin{pmatrix} \xi_\alpha \\ \chi^{\dagger\dot{\alpha}} \end{pmatrix}, \qquad I\!P_L\psi = \begin{pmatrix} \xi_\alpha \\ 0 \end{pmatrix}, \qquad I\!P_R\psi = \begin{pmatrix} 0 \\ \chi^{\dagger\dot{\alpha}} \end{pmatrix},$$

with the chiral projects $I\!P_{L/R} = \frac{1}{2}(1 \mp \gamma_5)$. Consequently, the adjoint bispinor is

$$\bar{\psi} = \psi^\dagger\gamma^0 = (\chi^\alpha, \xi^\dagger_{\dot{\alpha}})$$

and the charge-conjugate bispinor is

$$\psi^c \equiv -i\left(\bar{\psi}\gamma^0\gamma^2\right)^T$$

The indices are raised and lowered using the totally antisymmetric $\epsilon$-tensor

$$\xi^\alpha = \epsilon^{\alpha\beta}\xi_\beta, \quad \chi^{\dagger\dot{\alpha}} = \epsilon^{\dot{\alpha}\dot{\beta}}\chi^\dagger_{\dot{\beta}},$$

where

$$\epsilon^{\alpha\beta} = \epsilon^{\dot{\alpha}\dot{\beta}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = -\epsilon^{\beta\alpha} = \epsilon_{\beta\alpha} = -\epsilon_{\alpha\beta}.$$

We sum over undotted indices when $^\alpha \quad _\alpha$, we have

$$\xi\cdot\chi \equiv \xi^\alpha\chi_\alpha = \xi^\alpha\epsilon_{\alpha\beta}\chi^\beta = -\chi^\beta\epsilon_{\alpha\beta}\xi^\alpha = \chi^\beta\epsilon_{\beta\alpha}\xi^\alpha = \chi\cdot\xi.$$

Dotted indices are summed when $_{\dot{\alpha}} \quad ^{\dot{\alpha}}$

$$\xi^\dagger\cdot\chi^\dagger \equiv \xi^\dagger_{\dot{\alpha}}\chi^{\dagger\dot{\alpha}} = \xi^\dagger_{\dot{\alpha}}\epsilon^{\dot{\alpha}\dot{\beta}}\chi^\dagger_{\dot{\beta}} = -\chi^\dagger_{\dot{\beta}}\epsilon^{\dot{\alpha}\dot{\beta}}\xi^\dagger_{\dot{\alpha}} = \chi^\dagger_{\dot{\beta}}\epsilon^{\dot{\beta}\dot{\alpha}}\xi^\dagger_{\dot{\alpha}} = \chi^\dagger\cdot\xi^\dagger.$$

Weyl spinors and adjoint Weyl spinors are multiplied using the Pauli matrices:

$$\xi\cdot\sigma^\mu\cdot\chi^\dagger \equiv \xi^\alpha\sigma^\mu_{\alpha\dot{\alpha}}\chi^{\dagger\dot{\alpha}}, \qquad \chi^\dagger\cdot\bar{\sigma}^\mu\cdot\xi \equiv \chi^\dagger_{\dot{\alpha}}\bar{\sigma}^{\mu\dot{\alpha}\alpha}\xi_\alpha.$$

The Pauli matrices are related via

$$(\bar{\sigma}^\mu)^{\alpha\dot{\alpha}} = \epsilon^{\dot{\alpha}\dot{\beta}}\epsilon^{\alpha\beta}(\bar{\sigma}^\mu)_{\beta\dot{\alpha}} \tag{1}$$

Show that the following relations hold:

a) $\bar{\psi}_1 I\!P_L\psi_2 = \chi_1\cdot\xi_2$

b) $\bar{\psi}_1 I\!P_R\psi_2 = \xi_1^\dagger\cdot\chi_2^\dagger$

c) $\overline{\psi_1^c} I\!P_L\psi_2 = \xi_1\cdot\xi_2$

d) $\overline{\psi_1^c} I\!P_R\psi_2 = \chi_1^\dagger\cdot\chi_2^\dagger$

e) $\bar{\psi}_1\gamma^\mu I\!P_L\psi_2 = \xi_1^\dagger\cdot\bar{\sigma}^\mu\cdot\xi_2$

f) $\bar{\psi}_1\gamma^\mu I\!P_R\psi_2 = \chi_1\cdot\sigma^\mu\cdot\chi_2^\dagger = -\chi_2^\dagger\cdot\bar{\sigma}^\mu\cdot\chi_1$

g) $\mathcal{L}_{Dirac} = \bar{\psi}(i\partial\!\!\!/ - m)\psi = \xi^\dagger\cdot\bar{\sigma}^\mu\cdot i\partial_\mu\xi + \chi^\dagger\cdot\bar{\sigma}^\mu\cdot i\partial_\mu\chi - m(\chi\cdot\xi + \xi^\dagger\cdot\chi^\dagger) + \text{total derivative}$