III SUPERSYMMETRY

§1. SUSY Algebras und Multiplets

ϕ = complex scalar field

$\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi + \overline{\psi_L} i \not{\partial} \psi_L$

invar.: $\phi \to \phi + \overline{\Theta_L{}^c} \psi_L$

$\psi_L \to \psi_L - i \not{\partial} \Theta_L{}^c \phi$

[Θ: const. infinit. bispinor]

ψ = fermionic bispinor

conserved fermionic charge:

current: $j^\mu_\alpha = [(\not{\partial}\phi^*)\gamma^\mu \psi_L]_\alpha$

charge: $Q_\alpha = \int d^3\vec{x}\ j^0_\alpha(t, \vec{x})$

$\Longrightarrow \dot{Q}_\alpha = 0$

invariance also valid for interacting fields:

$$\begin{aligned} \mathcal{L} &= |\partial_\mu \phi|^2 + \overline{\psi_L} i \not{\partial} \psi_L - \left| m_S \phi + \frac{g_S}{2} \phi^2 \right|^2 \\ &- \frac{1}{2}(m_F + g_F \phi) \overline{\psi_L{}^c} \psi_L + h.c. \end{aligned}$$

symmetric under

$$\begin{aligned} \phi &\to \phi + \overline{\Theta_L{}^c} \psi_L \\ \psi_L &\to \psi_L - i \not{\partial} \Theta_L{}^c \phi - \left(m_S \phi + \frac{g_S}{2} \phi^2 \right) \Theta_L \end{aligned}$$

IF $g_S = g_F$ and $m_S = m_F$

SUSY: 1.) corresponding field d.o.f.
2.) identical masses
3.) identical couplings

Supersymmetry Algebra $N = 1$:

minimal symmetry realization
Q_α = lefthanded Weyl spinor: $\gamma_5 Q = -Q$

graded Lie algebra:

$$\begin{aligned}
\left\{Q_\alpha, Q_\beta\right\} &= \left\{\bar{Q}_\alpha, \bar{Q}_\beta\right\} = 0 \\
\left\{Q_\alpha, \bar{Q}_\beta\right\} &= 2\, \not{\mathbb{P}}_{L\alpha\beta} \\
[Q_\alpha, \mathbb{P}_\mu] &= 0 \\
[Q_\alpha, \mathbb{M}_{\mu\nu}] &= \frac{1}{2}(\sigma_{\mu\nu})_{\alpha\beta} Q_\beta
\end{aligned}$$

Poincaré algebra

spectrum: $[\mathbb{M}_{\mu\nu}, \bar{Q}_\alpha] = \frac{1}{2}\bar{Q}_\beta(\sigma_{\mu\nu})_{\alpha\beta}$: $M_{12} = J_3$

$\sigma_{12} = \sigma_3$

$$J_3 \bar{Q}_{1,2} = \bar{Q}_{1,2}\left(J_3 \pm \frac{1}{2}\right) \implies \begin{aligned}
&\bar{Q}_1 \text{ generates spin} = \frac{1}{2} \\
&Q_1 \text{ annihilates spin} = \frac{1}{2} \\
&\bar{Q}_2 \text{ annihilates spin} = \frac{1}{2} \\
&Q_2 \text{ generates spin} = \frac{1}{2}
\end{aligned}$$

Intermezzo: Poincaré algebra

$$\begin{aligned}
\mathbb{P}_\mu &= i\partial_\mu \\
\mathbb{M}_{\mu\nu} &= x_\mu \mathbb{P}_\nu - x_\nu \mathbb{P}_\mu + \Sigma_{\mu\nu} \\
\vec{J} &= (\mathbb{M}_{23}, \mathbb{M}_{31}, \mathbb{M}_{12}) \\
\vec{J}^2 &= L(L+1) \qquad (L = \ell + S)
\end{aligned}$$

$$\begin{aligned}
\text{e.g.} \quad \sigma_{\mu\nu} &= \frac{i}{2}[\gamma^\mu, \gamma^\nu] \qquad \leftarrow \text{Dirac spinors} \\
\Sigma_{\mu\nu} &= \frac{1}{2}\sigma_{\mu\nu} \\
\vec{\Sigma} &= \frac{1}{2}\begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix} \\
\vec{\Sigma}^2 &= \frac{3}{4} = S(S+1) \quad \Rightarrow \quad \text{spin} = \frac{1}{2}
\end{aligned}$$

[derived from $S(\Lambda)\psi(x') = \exp(i\vec{\epsilon}\vec{J})\ \psi(x)$ with infinitesimal rotations Λ and $\vec{x}' = \vec{x} - \vec{\epsilon} \times \vec{x}$]

Poincaré algebra:

$$\begin{aligned}
[\mathbb{M}_{\mu\nu}, \mathbb{M}_{\rho\sigma}] &= g_{\nu\rho}\mathbb{M}_{\mu\sigma} - g_{\mu\rho}\mathbb{M}_{\nu\sigma} - g_{\nu\sigma}\mathbb{M}_{\mu\rho} + g_{\mu\sigma}\mathbb{M}_{\nu\rho} \\
[\mathbb{M}_{\mu\nu}, \mathbb{P}_\rho] &= g_{\nu\rho}\mathbb{P}_\mu - g_{\mu\rho}\mathbb{P}_\nu \\
[\mathbb{P}_\mu, \mathbb{P}_\nu] &= 0
\end{aligned}$$

Multiplet Structure:

particle multiplet:	$\|\lambda\rangle, \left\|\lambda + \frac{1}{2}\right\rangle$
anti-particle:	$\|-\lambda\rangle, \left\|-\lambda - \frac{1}{2}\right\rangle$

Classification: $N = 1$ supersymmetry

supermultiplet	$\left[S, S + \frac{1}{2}\right]$
chiral	$\left[0, \frac{1}{2}\right]$
vector	$\left[\frac{1}{2}, 1\right]$
graviton	$\left[\frac{3}{2}, 2\right]$

generalization: $Q^{i=1,\ldots,N}_{\alpha=1,2}$:

$$\left\{Q^i_\alpha, Q^j_\beta\right\} = \left\{\bar{Q}^i_\alpha, \bar{Q}^j_\beta\right\} = 0$$
$$\left\{Q^i_\alpha, \bar{Q}^j_\beta\right\} = 2\delta_{ij}\, \not{P}_{L\alpha\beta} \qquad \cdots$$

maximum: $|-\lambda_{max}\rangle \cdots \left|-\lambda_{max} + \frac{N}{2}\right\rangle = |\lambda_{max}\rangle$

consist. FT: $\lambda_{max} \leq 2 \Longrightarrow N \leq 8$

Multiplet structure:

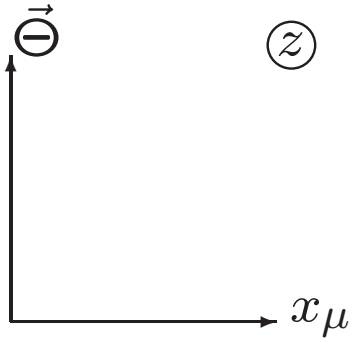
Theory	spin=0	spin=$\frac{1}{2}$	spin=1	spin=$\frac{3}{2}$	spin=2
N=1				1	1
N=2			1	2	1
N=3		1	3	3	1
N=4	2	4	6	4	1
N=5	10	11	10	5	1
N=6	30	26	16	6	1
N=7	70	56	28	8	1
N=8	70	56	28	8	1

particle content of multiplets in extended supergravity theories that contain a graviton

Theory	spin=0	spin=$\frac{1}{2}$	spin=1
N=1		1	1
N=2	2	2	1
N=3	6	4	1
N=4	6	4	1

gauge multiplets

Superspace:



$x = (x^0, x^1, x^2, x^3)$ $\quad z = (x; \Theta, \bar{\Theta})$

$\vec{\Theta} = (\Theta, \bar{\Theta})$ $\quad y = x + i\Theta\sigma\bar{\Theta}$

$\Theta = (\Theta_1, \Theta_2)$ $\quad \bar{\Theta} = (\bar{\Theta}_1, \bar{\Theta}_2)$

Grassmann numbers: $\left\{\Theta_i, \Theta_j\right\} = 0 \Rightarrow \Theta_i^2 = 0$

$f(\Theta) = f(0) + \Theta\ f'(0)$

$\dfrac{\partial}{\partial\Theta}\ f(\Theta) = f'(0)$

$\int d\Theta = 0 \qquad \int d\Theta\ \Theta = 1$

metric: $\Theta_\alpha, \bar{\Theta}_\alpha \quad (\alpha = 1, 2)$:

$\Theta^2 = \Theta_a\Theta^a = \epsilon^{ab}\Theta_a\Theta_b = 2\Theta_1\Theta_2$

$\quad = -\Theta^1\Theta_1 - \Theta^2\Theta_2 = \Theta_1\Theta^1 + \Theta_2\Theta^2$

$(\epsilon^{12} = -\epsilon^{21} = 1)$

SUPERFIELDS:

translation operator:

$$G(x;\Theta,\bar{\Theta}) = \exp\left[i\left(-x\mathbb{P} + \Theta Q + \bar{\Theta}\bar{Q}\right)\right]$$

$$G(0;\Theta',\bar{\Theta}')G(x;\Theta,\bar{\Theta}) = G(x'';\Theta'',\bar{\Theta}'')$$

$$\begin{aligned} \Theta'' &= \Theta + \Theta' \\ \bar{\Theta}'' &= \bar{\Theta} + \bar{\Theta}' \\ x'' &= x - i\left[\Theta'\sigma\bar{\Theta} - \Theta\sigma\bar{\Theta}'\right] \end{aligned} \qquad \begin{aligned} \sigma &= (\mathbb{1}, \vec{\sigma}) \\ \bar{\sigma} &= (\mathbb{1}, -\vec{\sigma}) \end{aligned}$$

differential representation:

$$\begin{aligned} \mathbb{P} &= i\partial \\ Q &= \frac{\partial}{\partial\Theta} - i\sigma\bar{\Theta}\partial \\ \bar{Q} &= \frac{\partial}{\partial\bar{\Theta}} - i\Theta\sigma\partial \end{aligned}$$

superfield: combination of fields belonging to the same supermultiplet to a unified field

general [scalar] superfield with spin ≤ 1:

$$\begin{aligned} \Phi(x;\Theta,\bar{\Theta}) &= \phi(x) + \Theta\psi(x) + \bar{\Theta}\bar{\chi}(x) + \Theta^2 F(x) \\ &+ \bar{\Theta}^2 N(x) + \Theta\sigma_\mu\bar{\Theta}V^\mu(x) + \Theta^2\bar{\Theta}\bar{\lambda}(x) \\ &+ \bar{\Theta}^2\Theta\alpha(x) + \Theta^2\bar{\Theta}^2 D(x) \end{aligned}$$

16 real bosonic fields: ϕ, F, N, V_μ, D
16 fermionic fields: $\psi, \bar{\chi}, \bar{\lambda}, \alpha$

powers of superfields are again superfields
Φ reducible [spin $= 0, \frac{1}{2}, 1$]: irreducible parts $\rightarrow$

$$\begin{aligned}
\text{SUSY–Trafo:}\Phi(x;\Theta;\bar{\Theta}) &\rightarrow G(0;\epsilon;\bar{\epsilon})\Phi(x;\Theta;\bar{\Theta}) \\
&= \Phi(x+\xi;\Theta+\epsilon,\bar{\Theta}+\bar{\epsilon}) \\
&\quad (\xi = -i[\epsilon\sigma\bar{\Theta} - \Theta\sigma\bar{\epsilon}]) \\
&= \Phi(x;\Theta;\bar{\Theta}) + \delta\Phi
\end{aligned}$$

$$\delta\Phi = \xi\partial\Phi + \epsilon\frac{\partial\Phi}{\partial\Theta} + \bar{\epsilon}\frac{\partial\Phi}{\partial\bar{\Theta}} = \delta\phi(x) + \Theta\delta\psi(x) + \cdots$$

$$\begin{aligned}
\delta\phi(x) &= \epsilon\psi(x) + \bar{\epsilon}\bar{\chi}(x) \\
\delta\psi(x) &= \sigma\bar{\epsilon}[i\partial\phi(x) + V(x)] + 2\epsilon F(x) \\
\delta\bar{\chi}(x) &= \bar{\sigma}\epsilon[i\partial\phi(x) - V(x)] + 2\bar{\epsilon}N(x) \\
\delta F(x) &= -\frac{i}{2}\partial\psi(x)\sigma\bar{\epsilon} + \bar{\epsilon}\bar{\lambda}(x) \\
\delta N(x) &= \frac{i}{2}\epsilon\sigma\partial\bar{\chi}(x) + \epsilon\alpha(x) \\
\delta V_\mu(x) &= \epsilon\sigma_\mu\bar{\lambda}(x) + \alpha(x)\sigma_\mu\bar{\epsilon} \\
&\quad + \frac{i}{2}\partial^\nu[\epsilon\sigma_\nu\bar{\sigma}_\mu\phi(x) - \bar{\chi}\bar{\sigma}_\mu\sigma_\nu\bar{\epsilon}] \\
\delta\bar{\lambda}(x) &= \bar{\epsilon}\left[2D(x) + \frac{i}{2}\partial V(x)\right] + i\bar{\sigma}^\mu\epsilon\partial F(x) \\
\delta\alpha(x) &= \epsilon\left[2D(x) - \frac{i}{2}\partial V(x)\right] + i\sigma\bar{\epsilon}\partial N(x) \\
\delta D(x) &= \frac{i}{2}\epsilon\sigma\partial\bar{\lambda} - \frac{i}{2}\partial\alpha(x)\sigma\bar{\epsilon}
\end{aligned}$$

chiral superfield: ϕ, ψ vectorial field: V_μ, λ

CHIRAL SUPERFIELD

side conditions: projection on irreducible ingredients

$\bar{D} = \dfrac{\partial}{\partial\bar{\Theta}} + i\Theta\sigma\partial$: SUSY-invariant $\{\bar{D}, Q\} = \cdots = 0$

chiral superfield: $\bar{D}\Phi_- = 0$

antichiral superfield:

$D\Phi_+ = 0$ $\qquad\qquad$ $D = \dfrac{\partial}{\partial\Theta} + i\sigma\bar{\Theta}\partial$

solution:

$\Phi_- = \Phi(x + i\Theta\sigma\bar{\Theta}; \Theta, 0)$ is general chiral superfield

$\bar{D}\Phi_- = -i\Theta\sigma\partial\Phi + i\Theta\sigma\partial\Phi = 0$

$\Phi_-(y) = \phi(y) + \sqrt{2}\Theta\psi(y) + \Theta^2 F(y)$

$[y = x + i\Theta\sigma\bar{\Theta}]$

$\delta\phi = \sqrt{2}\epsilon\psi$ $\delta\psi = \sqrt{2}[i\sigma\bar{\epsilon}\partial\phi + \epsilon F]$ $\delta F = -\sqrt{2}i\partial\psi\sigma\bar{\epsilon}$

$[y \to y + 2i\Theta\sigma\bar{\epsilon}, \Theta \to \Theta + \epsilon, \bar{\Theta} \to \bar{\Theta} + \bar{\epsilon}]$

[F = non–propagating auxiliary field]

VECTOR–SUPERFIELD

superfield = hermitesch: $V^\dagger = V$

gauge trafo: $V \to V + (\chi + \chi^\dagger) \to$ minimal field content:

Wess–Zumino gauge

[$\chi^{(\dagger)}$ = (anti)chiral superfield: $\Phi \to e^{-g\chi}\Phi$, $\Phi^\dagger \to \Phi^\dagger e^{-g\chi^\dagger}$]

$$\boxed{\begin{array}{rcl} V & = & -\Theta\sigma_\mu\bar{\Theta}A^\mu(x) + i\Theta^2\bar{\Theta}\bar{\lambda}(x) \\ & - & i\bar{\Theta}^2\Theta\lambda(x) + \Theta^2\bar{\Theta}^2 D(x) \end{array}}$$

$$\begin{array}{rcl} A_\mu(x) & = & \text{vector field} \\ \lambda(x) & = & \text{vectorino field} \\ D(x) & = & \text{auxiliary field [non–propagating]} \end{array}$$

field strength:

$$\begin{array}{rcl} V_a & = & -\dfrac{1}{4}\bar{D}^2 D_a V \\ & = & i\lambda_a - \Theta_b \left[2\delta_a{}^b D + (\sigma_{\mu\nu})_a{}^b F^{\mu\nu}\right] \\ & - & \Theta^2(\sigma\partial)_{a\dot{a}}\bar{\lambda}^{\dot{a}}\Big|_{y=x+i\Theta\sigma\bar{\Theta}} \end{array}$$

$\bar{D}^3 = D^3 = 0 \quad \Rightarrow \bar{D}V_a = 0$ (chiral superfield)

ACTION:

SUSY and [$U(1)$–]gauge invariant:

$$
\begin{aligned}
S &= S_V + S_m + S_W : \\
S_V &= \frac{1}{4}\int d^6z\ V_\alpha V^\alpha + h.c. \qquad \text{gauge field part} \\
S_m &= \int d^8z\ \Phi^\dagger e^{gV}\Phi \qquad \text{matter + gauge int.} \\
S_W &= \int d^6z\ W[\Phi] + h.c. \qquad \underline{\text{superpotential:}} \\
& \qquad\qquad W = \frac{m}{2}\Phi^2 + \frac{\lambda}{6}\Phi^3 \\
& \qquad\qquad\qquad \uparrow \text{masses}
\end{aligned}
$$

$$[d^8z = d^4x\ d^2\Theta\ d^2\bar{\Theta},\ d^6z = d^4x\ d^2\Theta \to d^4x]$$

LAGRANGE DENSITY

$$
\begin{aligned}
\mathcal{L} &= \mathcal{L}_V + \mathcal{L}_m + \mathcal{L}_W : \\
\mathcal{L}_V &= -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}\bar{\lambda}i\not{\partial}\lambda + \frac{1}{2}D^2 \qquad \rightsquigarrow \underline{\underline{D^2}} \\
\mathcal{L}_m &= |D_\mu\phi|^2 + \bar{\psi}_L i\not{D}\psi_L + i\sqrt{2}g[\phi^*\overline{\psi_L{}^c}\lambda_L - \phi\overline{\lambda_L}\psi_L{}^c] \\
&+ g|\phi|^2 D + |F|^2 \\
\mathcal{L}_W &= -\frac{1}{2}\frac{\partial^2 W}{\partial\phi\partial\phi}\overline{\psi_L{}^c}\psi_L + F\frac{\partial W}{\partial\phi} + h.c. \qquad \rightsquigarrow \underline{\underline{|F|^2}}
\end{aligned}
$$

$$\underline{\underline{V = \frac{1}{2}D^2 + |F|^2}}$$

1.) $D(x), F(x)$ are non–propagating auxiliary fields:

$$\frac{\partial \mathcal{L}}{\partial D} = \frac{\partial \mathcal{L}}{\partial F} = \frac{\partial \mathcal{L}}{\partial F^*} = 0$$

$D = -g|\phi|^2$

$$\boxed{\mathcal{L}_D = -\frac{g^2}{2}|\phi|^4}$$

quartic coupling of scalar field is
fixed by gauge coupling.

$\Longrightarrow$ MSSM: Higgs ϕ^4 coupling determined solely by gauge int.

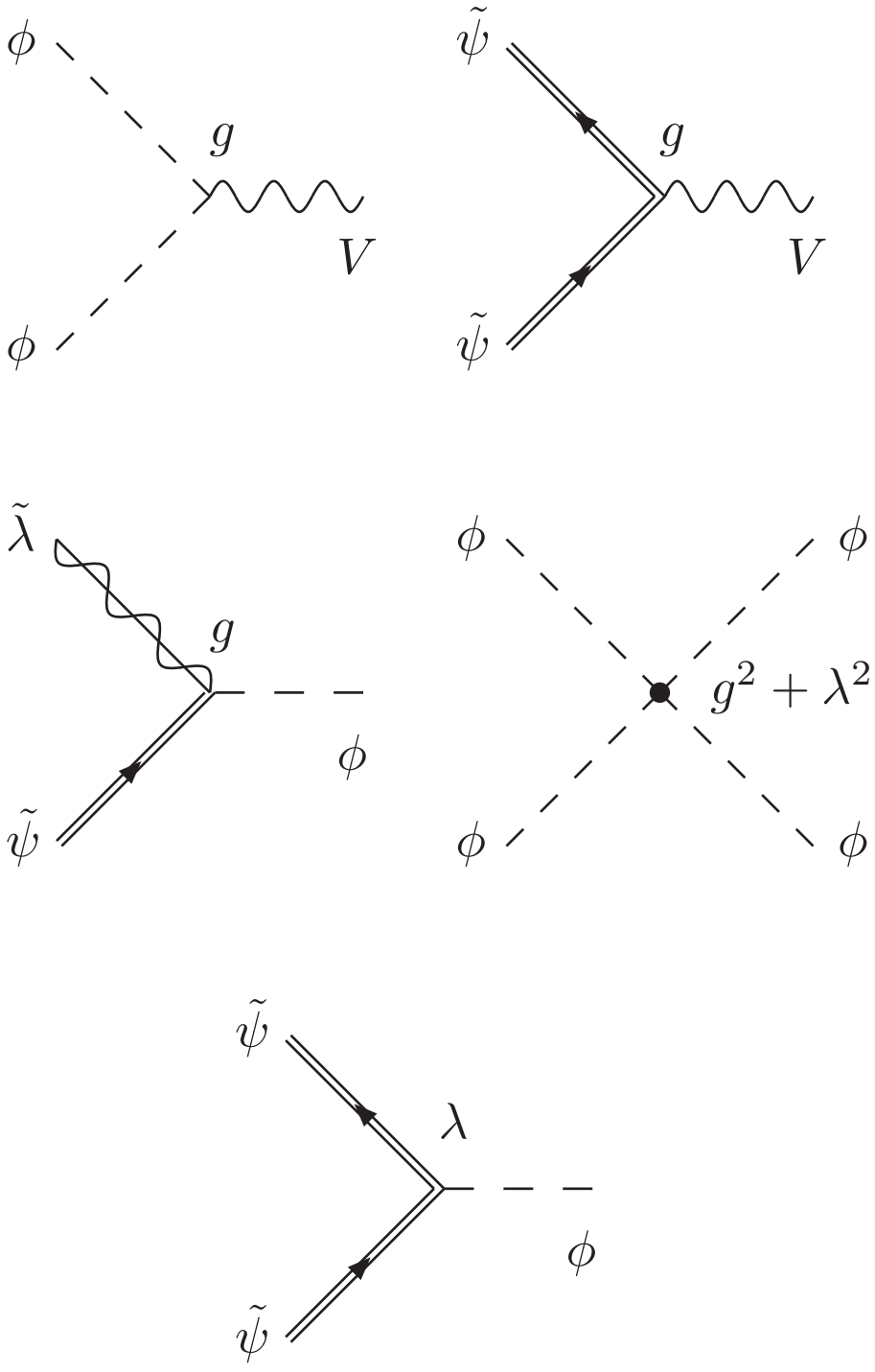
$$F^* = -\frac{\partial W}{\partial \phi} \qquad\qquad F = -\frac{\partial W^*}{\partial \phi^*}$$

2.) supersymmetric particles interact in pairs. Multiplicative quantum number:

$R = (-1)^{3(B-L)+2S}$

R–parity	=	$+1$ for SM particles
	=	-1 for superpartners

$\Longrightarrow$ production in pairs, LSP stable


ϕ
g
V
ϕ
$\tilde{\psi}$
g
V
$\tilde{\psi}$
$\tilde{\lambda}$
g
ϕ
$\tilde{\psi}$
ϕ
ϕ
$g^2 + \lambda^2$
ϕ
ϕ
$\tilde{\psi}$
λ
ϕ
$\tilde{\psi}$